MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development Materials

3 Multiplicative reasoning

Theme overview

Guidance document | Key Stage 3

Making connections

'Teaching for mastery' describes the elements of classroom practice and school organisation that combine to give students the best chance of developing a deep, connected, embedded and sustainable understanding of mathematics.

At any one point in a student's journey through school, achieving mastery means acquiring a secure understanding of the mathematics that has been taught to enable them to move on to more advanced material.

To achieve this, students need to understand the interconnected nature of mathematics and how one idea builds on and develops from other ideas. To this end, the NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of connected ideas or 'core concepts'.

The theme *Multiplicative reasoning* covers the following interconnected core concepts:

3.1 Understanding multiplicative relationships

3.2 Trigonometry

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Why is this mathematical theme important?

Students may have met the idea of multiplication in primary schools by thinking about equal groupings of objects and counting on in equal amounts from zero. If your students have followed the NCETM Primary Mastery Materials, then they will also have worked, from early on, with the idea of multiplication as scaling. This develops into the idea that the product of two numbers can be greater or less than both numbers depending on whether they are greater than or less than one. As the numbers with which students work extend beyond integers to include fractions and decimals, the scaling view of multiplication becomes increasingly important.

Thinking of multiplication as scaling leads to the important idea that a number can be transformed into another number both additively and multiplicatively (e.g. 4 can become 6 by adding 2 or by multiplying by 1.5). Multiplicative transformations give rise to ideas of ratio, proportionality, percentage increase and decrease, rates of change, enlargement and similarity, and trigonometric ratios. In fact, when the Key Stage 3 and 4 curricula are examined closely, it becomes clear that proportional reasoning underpins so many ideas and that students' understanding and facility in this key area supports a connected view of much of the secondary curriculum. It allows students to link what might be seen as separate, isolated topics, each with their own different set of rules and techniques, and to see them as examples of the same fundamental mathematical idea.

Key underpinning knowledge

Several important considerations are key to students gaining a secure and deep understanding of the mathematics within this theme, namely:

- that any two numbers can be connected and compared using a multiplier
- that fractions, ratios and percentages are all different ways of expressing multiplicative relationships and operations
- that certain models, images and representations are important and can support thinking and reasoning about multiplicative relationships
- that compound measures, such as speed (e.g. km/h or mph), pressure (e.g. N/m^2 or psi – pounds per square inch), density (mass per unit area), and so on, are all ways of describing the multiplicative relationship between one measure and another in certain situations.

Statements of knowledge, skills and understanding

Each of the two core concepts within the theme *Multiplicative reasoning* has been broken down further into a set of statements of knowledge, skills and understanding, as listed below.

3.1 Understanding multiplicative relationships

3.1.1 Understand the concept of multiplicative relationships

3.1.2 Understand that multiplicative relationships can be represented in a number of ways and connect and move between those different representations

3.1.3 Understand that fractions are an example of a multiplicative relationship and apply this understanding to a range of contexts

3.1.4 Understand that ratios are an example of a multiplicative relationship and apply this understanding to a range of contexts

3.1.5 Understand that percentages are an example of a multiplicative relationship and apply this understanding to a range of contexts

3.1.6 Understand proportionality

3.2 Trigonometry

3.2.1 Understand the trigonometric functions

3.2.2 Use trigonometry to solve problems in a range of contexts

We have produced guidance documents that offer an overview of each core concept, as well as an overview of the content of each statement of knowledge, skills and understanding. We have also broken down each of the latter into a series of key ideas to support planning, with some of the key ideas exemplified as to what teaching for mastery may look like.

We make no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions we offer in these key ideas are intended to help you think about the learning journey irrespective of the number of lessons taught.

Not all key ideas are of equal weight and the amount of classroom time required for them to be mastered will vary, but each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

These materials are designed for teachers to use collaboratively when planning how they will teach for a secure and deep understanding of mathematics throughout Key Stage 3. They are underpinned by a clear set of pedagogical principles and practices.

The *Multiplicative reasoning* core concept guidance documents[1] can be downloaded from the NCETM website.

Links to the national curriculum

A mapping[2] of all statements of knowledge skills and understanding to the national curriculum Key Stage 3 programme of study is available on the NCETM website.

Previous learning

From Upper Key Stage 2, students will bring experience of:

- multiplying proper fractions and mixed numbers by whole numbers, supported by materials and diagrams
- recognising the per cent symbol (%) and understanding that per cent relates to 'number of parts per hundred', and writing percentages as a fraction with denominator 100, and as a decimal
- using all four operations to solve problems involving measure (for example, length, mass, volume, money) using decimal notation, including scaling
- solving problems involving the relative sizes of two quantities where missing values can be found by using integer multiplication and division facts
- solving problems involving the calculation of percentages (for example, of measures, and such as 15% of 360) and the use of percentages for comparison
- solving problems involving similar shapes where the scale factor is known or can be found
- solving problems involving unequal sharing and grouping, using knowledge of fractions and multiples.

Future learning

In Key Stage 4, students will build on the core concepts in this mathematical theme to:

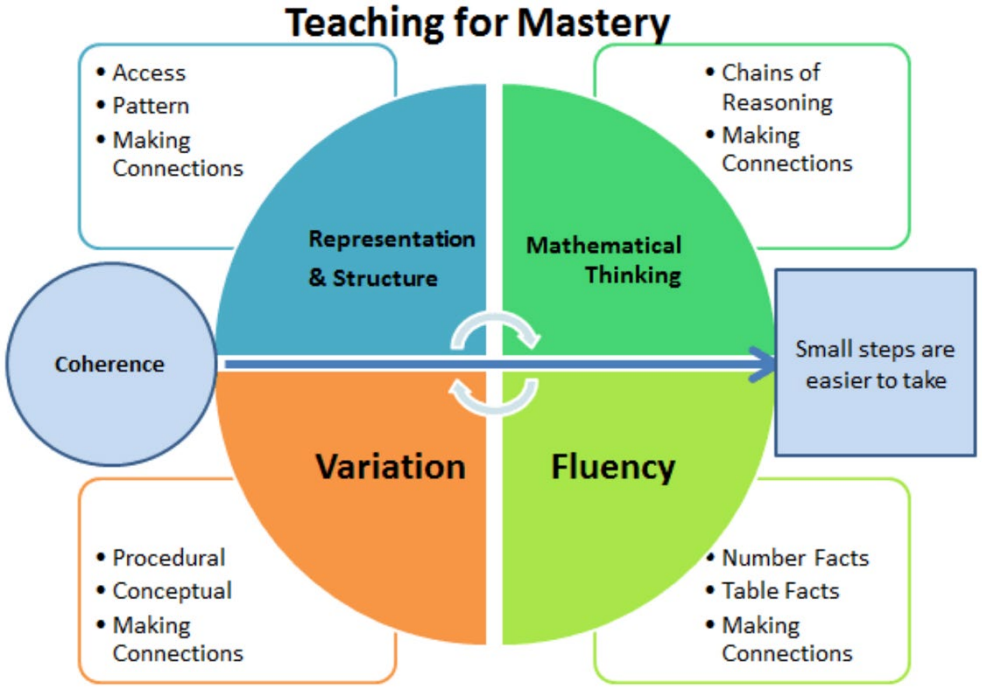
- compare lengths, areas and volumes using ratio notation and/or scale factors; make links to similarity (including trigonometric ratios)
- convert between related compound units (speed, rates of pay, prices, density, pressure) in numerical and algebraic contexts
- understand that X is inversely proportional to Y is equivalent to X is proportional to $\frac{1}{Y}$
- {construct and} interpret equations that describe direct and inverse proportion
- interpret the gradient of a straight-line graph as a rate of change; recognise and interpret graphs that illustrate direct and inverse proportion
- {interpret the gradient at a point on a curve as the instantaneous rate of change; apply the concepts of instantaneous and average rate of change (gradients of tangents and chords) in numerical, algebraic and graphical contexts}
- set up, solve and interpret the answers in growth and decay problems, including compound interest {and work with general iterative processes}.

Please note: Braces { } indicate additional mathematical content to be taught to more highly attaining students.

Teaching for mastery

A central component in the NCETM/Maths Hubs programmes to support the development of teaching for mastery has been discussion of Five Big Ideas[3] underpinning teaching for mastery. These are:

- Coherence
- Representation and structure
- Variation
- Fluency
- Mathematical thinking

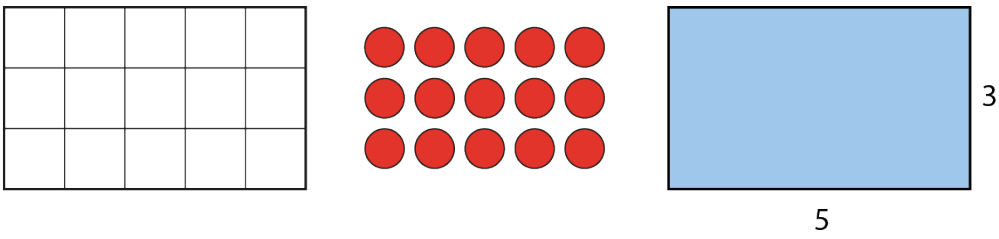
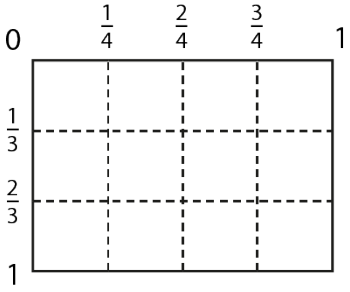


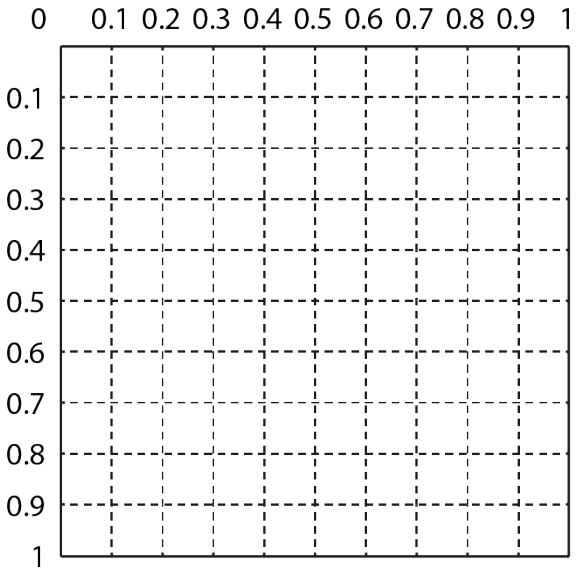
The sections below offer guidance about how these ideas relate to *Multiplicative reasoning*.

Coherence

It is important to find a balance between focusing on important elements of this theme where it is useful to plan a coherent set of small steps and not conflate too many ideas (for example, when introducing the idea of ratio, separating out dividing a whole into parts, finding the whole given a part and the ratio, finding a part given the other part and the ratio, etc.) and appreciating how each idea is connected to others in the theme. For example, using fractions, percentages and ratio as ways of exploring the same problem and seeing compound measures as an example of ratio.

Representation and structure

Representations	Structural understanding
Arrays and other area models for multiplication	 When the rectangle has continuous measures for the dimensions, it becomes a useful way of thinking about the product of any two numbers including decimals and fractions. Arrays of dots, as shown here, offer a useful image to understand the structure of integer multiplication but are less adaptable for use with products that are not integers. The diagram below helps students to make sense of the various calculations: $\frac{1}{4}\times\frac{1}{3}$, $\frac{1}{4}\times\frac{2}{3}$, $\frac{3}{4}\times\frac{2}{3}$, etc.  Similarly, the following diagram supports understanding of why 0.1s multiplied by 0.1s gives 0.01s and how the product 3×7 is connected to 0.3×0.7:

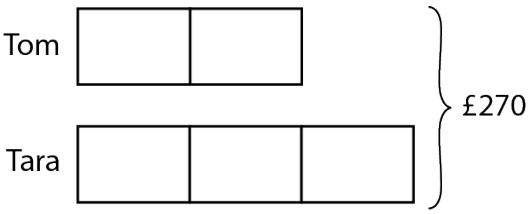


Bar models

Bar models can be very useful to support students in representing (literally, re-presenting) problems in order to reveal multiplicative structures (as appropriate) and thus become aware of how to proceed to a solution.

For example:

Tom and Tara share £270 between them in the ratio 2 : 3. How much do they each receive?



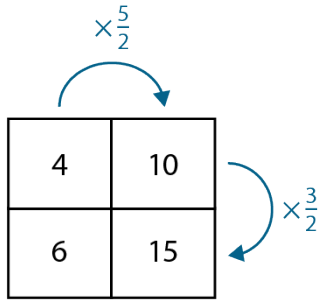
Ratio table and double number line (also known as 'stacked number lines')

The comparison of corresponding entries in two different sets of multiples, such as from a multiplication table like this:

2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30

can be a useful context in which to explore multiplicative relationships.

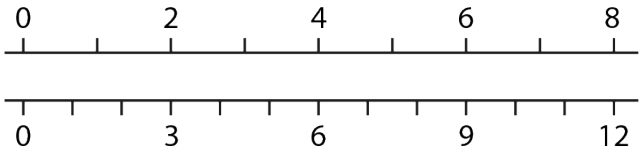
Taking any two rows from such an arrangement, like this:



results in a ratio table which succinctly encapsulates the given multiplicative relationship or ratio both within, as well as between, the pairs of numbers. Furthermore, the ratio table can be used to calculate missing values in a multiplicative relationship, supporting students in identifying the most efficient multiplier. For example:

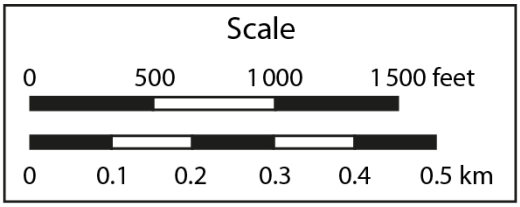
4	7
6	?

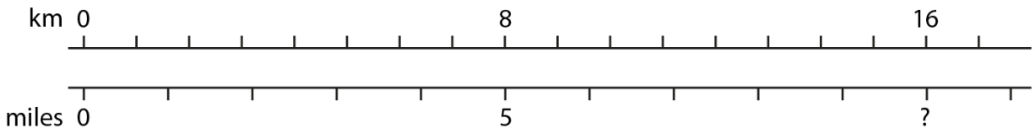
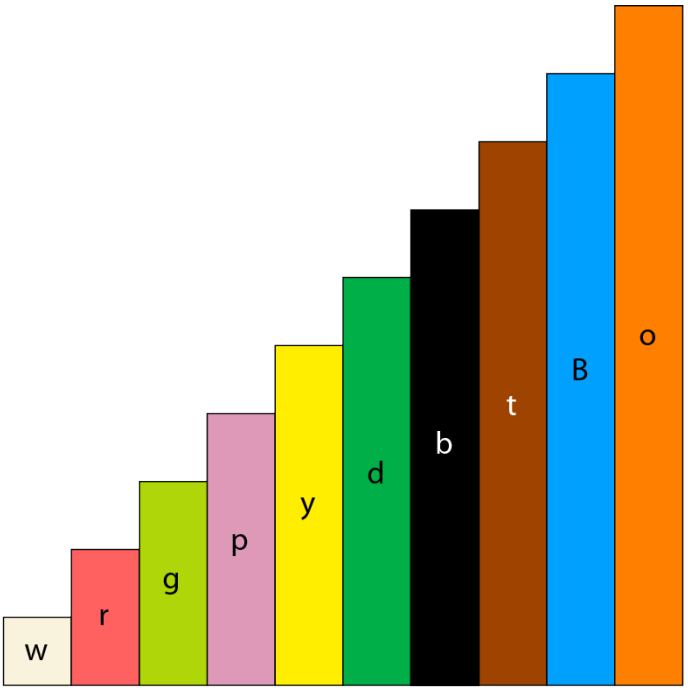
When, through discussion and exploration, it becomes clear that such relationships also exist for interim values, then the double number line becomes a useful image of the multiplicative relationship, as well as offering a ready reckoner for students to estimate solutions to multiplicative problems before calculating accurately.

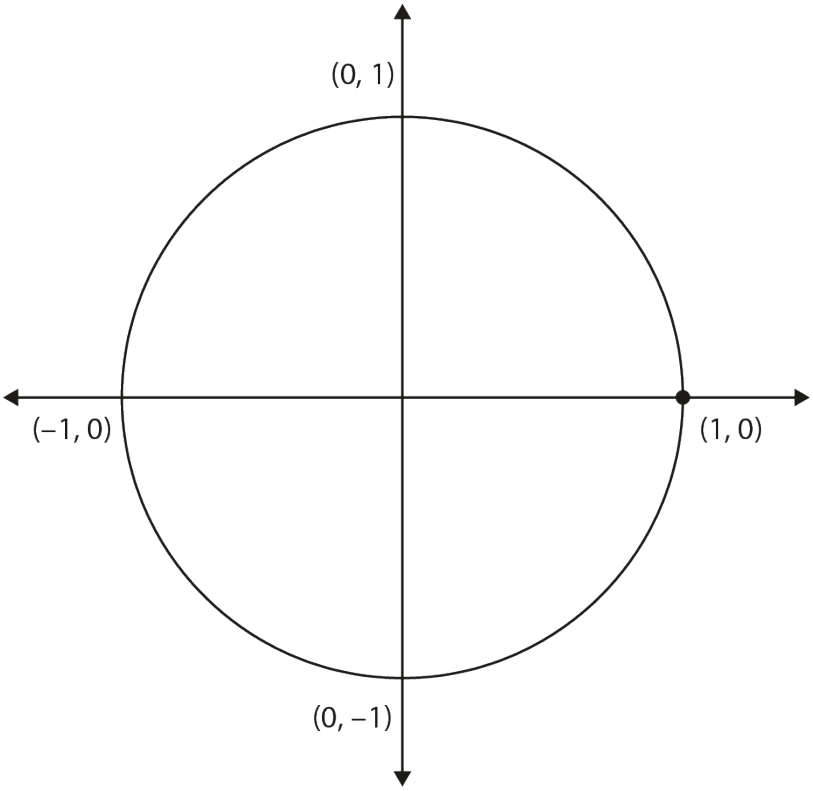
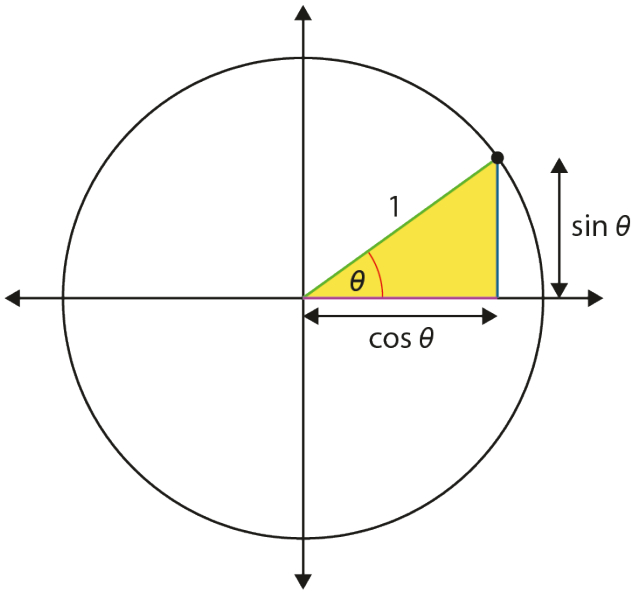


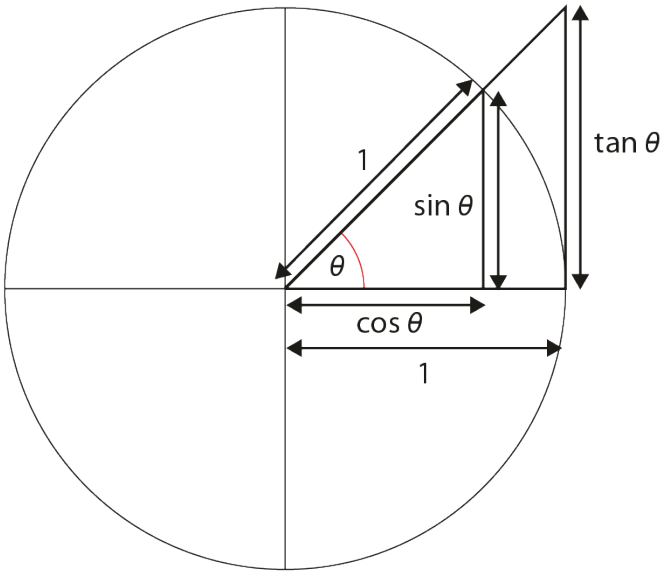
As well as allowing students to see scaling along the line, the double number line also exposes the constant relationship between the upper and lower lines (in this case $\frac{2}{3}$).

Students may already be familiar with double number lines in the context of map scales. For example:



	This familiar context may support students in making sense of multiplicative relationships, for example, when converting between miles and kilometres.  Here, students can see both the scaling relationship (working along the lines, $8 \times 2 = 16$, so the missing value must be $5 \times 2 = 10$) and the constant multiplier between the two lines ($16 \times \frac{5}{8} = 10$). The real power of the double number line is not merely as an efficient means of calculation, but as a representation which reveals and clarifies the multiplicative relationship.
Cuisenaire® rods	 These coloured rods of varying sizes from 1 cm to 10 cm long are a powerful representation to support students' understanding of multiplicative relationships. What is important is to highlight the relationships between the rods and not to impose fixed values on them (i.e. the red rod is not always '2' but it is always two-thirds of the light green rod; two-fifths of the yellow rod, etc.). By allowing the value of any particular rod to vary, these multiplicative relationships can be explored. For example, asking students to find the value of all the other rods in the box if the red rod has a value of one-third, forces students to think multiplicatively and to become familiar with the language, notation and meaning of fractions.

The unit circle	A point moving around a unit circle (a circle of radius 1) provides a very rich representation for understanding and visualising the values of the trigonometric (or circular) functions.  When it is placed on a coordinate grid with its centre at the origin, the cosine and sine functions are represented by the *x*- and *y*-coordinates, respectively, of the point for different values of the angle θ. 

	From the triangles identified inside the unit circle the meaning of sin θ, cos θ and tan θ and the relationships between them can be seen:  By using proportional reasoning, these trigonometric ratios for any right-angled triangle that is an enlargement of these 'unit' triangles can be inferred.

Further guidance on using representations[4] in Key Stage 3 is available on the NCETM website.

Variation

Three aspects of variation that can be usefully employed:

1. Careful **choice of exercises** to 'home in' on the important concept. For example, when working on multiplying by a fraction, keeping one number the same and only varying the fraction multiplier (including those greater than as well as less than one) can helpfully draw attention to multiplication as scaling and under what conditions multiplication makes larger or smaller.
2. Careful **choice of examples** to include *'what it is'* (using non-standard as well as standard examples) and *'what it is not'*. For example, when offering examples of pairs of similar shapes, choose ones that look similar but are not, because addition (rather than multiplication) has been used to increase corresponding sides.
3. Rather than focusing on the answer and asking only that students solve a problem, inviting students to see **in how many different ways they can solve a problem** can prompt important discussions about methods and processes, and support students' development of increasingly efficient, creative and elegant approaches. For example, when using 5 miles ≈ 8 km to convert 11 miles to kilometres, students might use the scalar multiplier, the functional multiplier, or a combination of multiplicative and additive approaches. Discussing which of these strategies is most efficient, and which can be generalised for any multiplicative relationship, may be fruitful and will support students' development of increasingly efficient, creative and elegant approaches.

Fluency

A key aspect of fluency is the ability to choose the most efficient strategy for a problem or calculation. Seeing percentage change, for example, as a wholly multiplicative process (i.e. increasing an amount by 15% is equivalent to multiplying by 1.15) enables students to calculate both direct and reverse percentage problems efficiently and accurately.

Mathematical thinking

Throughout all the work that falls within *Multiplicative reasoning*, the emphasis is on understanding the structure of multiplication and of going beyond finding the answer to problems to reason why methods work. It is vital that students are prompted to reason, explain, conjecture and prove through carefully planned teacher–student and student–student discussion and not merely to listen to and follow carefully constructed teacher demonstrations and explanations. For example, when learning about percentage increase and decrease, the following question could be discussed and explored:

> *'Increase a sum of money by 50% and then decrease this new amount by 50%.*
> *Can you explain why you don't end up with the original amount?*
> *Can you draw a diagram to show this?'*

Further reading

NCETM secondary assessment materials[5]

Exemplar questions, tasks and activities, which may be used to support teaching and assessment. The assessment materials are mapped against the key mathematical skills and concepts within the national curriculum Key Stage 3 programme of study. Of particular relevance to *Multiplicative reasoning* is the section focusing on ratio, proportion and rates of change (pages 28–34).

Weblinks

1 Theme 3: *Multiplicative reasoning* – core concept guidance documents
https://www.ncetm.org.uk/resources/53532

2 NCETM Key Stage 3 mastery curriculum structure, including national curriculum mapping
https://www.ncetm.org.uk/secondarymasterypd#curriculum_structure

3 Five Big Ideas in Teaching for Mastery
https://www.ncetm.org.uk/resources/50042

4 Representations in Key Stage 3 – guidance documents
https://www.ncetm.org.uk/resources/53609

5 NCETM secondary assessment materials
https://www.ncetm.org.uk/resources/51246