MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Multiplication and Division

2.5 Commutativity (part 2), doubling and halving

Teacher guide | Year 2

Teaching point 1:

The same multiplication equation can have two different grouping interpretations. Problems about two/five/ten equal groups can be solved using facts from the two/five/ten times table. (commutativity)

Teaching point 2:

If two is a factor, knowledge of doubling facts can be used to find the product; problems about doubling can be solved using facts from the two times table.

Teaching point 3:

Halving is the inverse of doubling; problems about halving can be solved using facts from the two times table and known doubling facts.

Teaching point 4:

Products in the ten times table are double the products in the five times table; products in the five times table are half of the products in the ten times table.

Overview of learning

In this segment children will:

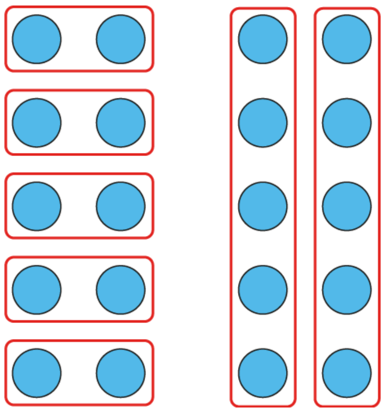
- explore, through the use of arrays and within the context of the two, five and ten times tables, how one multiplication equation can correspond to two different grouping interpretations; for example, $5 \times 2 = 10$ can represent five groups of two or two groups of five (five, two times)
- use their knowledge of two, five and ten times table facts (building on the previous point) to solve problems about two/five/ten equal groups, contrasting this with solving problems about groups of two/five/ten
- extend their understanding of commutativity to situations where two, five or ten are not one of the factors; for example, 6×3 could represent six groups of three or three groups of six (six, three times)
- build on knowledge of commutativity to connect the two times table to doubling/halving strategies and problems
- apply their understanding of doubling and halving to make connections between the five and ten times tables.

By this stage in the spine, children should be fluent with their two, five and ten times table facts. They should also be confident in writing factors in either order to represent groups of two, five or ten. For example, children can write two equations to represent three groups of two:

- $3 \times 2 = 6$ *'Three twos are six.'*
- $2 \times 3 = 6$ *'Two, three times is six.'*

In segment *2.3 Times tables: groups of 2 and commutativity (part 1)*, this was described as the 'one interpretation, two equations' understanding of commutativity.

In this segment (*Teaching point 1*), children will explore how a single multiplication equation can represent two different grouping interpretations; we will refer to this as the 'one equation, two interpretations' understanding of commutativity. The concept will be introduced using familiar language alongside array representations. For example, the equation $5 \times 2 = 10$ can represent five groups of two, or five, two times, with both being revealed by the same array:



Working with known times tables (two, five and ten), children will practise interpreting equations and arrays in this way. They will then apply the learning to solve contextual problems about two/five/ten equal groups using their known multiplication facts. For example, *'There are two boxes of cakes. Each box contains four cakes. How many cakes are there altogether?'*

At first glance, this question appears to require a times table that children don't yet know (the four times table, since the question is about groups of four). However, the other factor is two; the question is about two equal groups, so children can use the two times table to solve the problem:

- *'Two groups of four is equal to two, four times.'*
- *'I know that two, four times is equal to eight, so I know that two groups of four is equal to eight.'*

Once children have mastered the steps in *Teaching point 1*, they will connect the idea of two equal groups to their knowledge of doubling and halving. In *Spine 1: Number, Addition and Subtraction*, segment *1.7*, doubling was explored in terms of additive reasoning, i.e. *'When both addends are the same, we are doubling.'* In *Teaching point 2,* once children see that doubling is structurally equivalent to two equal groups, they will practise using known two times table facts to solve problems about 'doubling' or 'twice as many/much'. They will then extend their doubling facts and strategies and practise using those to solve problems about two equal groups (including going beyond two groups of 12, i.e. beyond the known two times table).

Children have already learnt that halving is the inverse of doubling (*Spine 1*, segment *1.7*). *Teaching point 3* follows a similar progression to *Teaching point 2*, now looking at halving, but using similar representations to support children in making connections with doubling. Note that children will essentially be dividing by two, using multiplication or doubling facts, but they will not be using the word 'division' or the associated symbol. Halving is linked to division with a divisor of two in segment *2.6 Structures: quotitive and partitive division*, *Teaching point 4*.

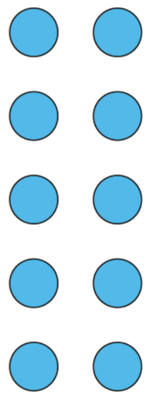
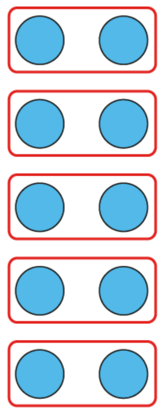
The final teaching point in this segment brings together the connections that children already spotted between the five and ten times table (segment *2.4 Times tables: groups of 10 and of 5, and factors of 0 and 1*) and the language of doubling and halving. In future times table segments, similar language will be used to describe the relationship between times tables. For example, in segment *2.7 Times tables: 2, 4 and 8, and the relationship between them*, children will learn that products in the four times table are *double* the products in the two times table, and, conversely, that products in the two times table are *half* of the products in the four times table.

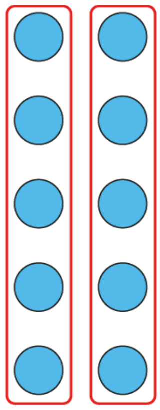
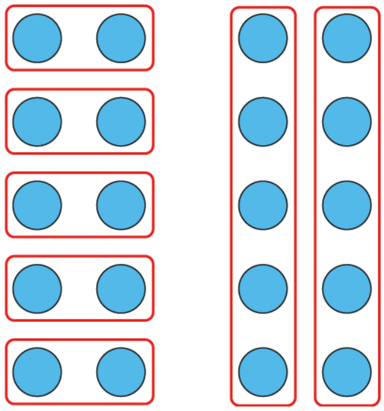
An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

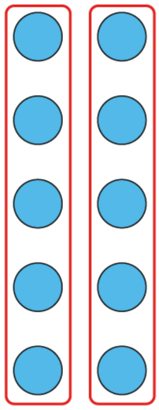
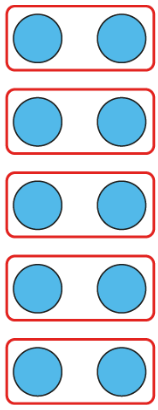
Teaching point 1:

The same multiplication equation can have two different grouping interpretations. Problems about two/five/ten equal groups can be solved using facts from the two/five/ten times table. (commutativity)

Steps in learning

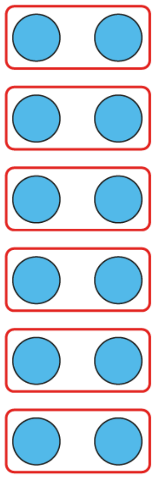
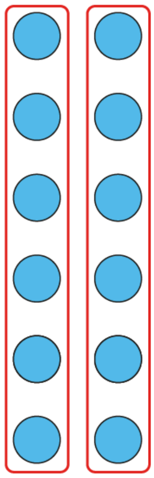
	Guidance	Representations
1:1	In segment *2.3 Times tables: groups of 2 and commutativity (part 1)*, children learnt how one grouping context can be represented by two multiplication equations. For example, three groups of two can be represented by: • $3 \times 2 = 6$ *'Three twos are six.'* • $2 \times 3 = 6$ *'Two, three times is six.'* Now move on to look at how a single multiplication equation can have two different grouping interpretations (with the product remaining the same). For example, *5 × 2* can represent either five groups of two, or two groups of five. Present a two-by-five array and ask children: • *'How many groups of two are there?'* • *'How can we represent this with a multiplication equation?'* Draw around the groups of two and focus on the equation with '2' as the second factor. Use the language *'five groups of two is equal to ten'* and *'five times two is equal to ten'* to describe the situation. Then, keeping the equation visible, remove the circles around the groups of two and ask children *'How many groups of five are there?'* Draw around the groups of five and look again at the equation *5 × 2 = 10*. Now use the language *'five two times is equal to ten'* and *'five times two is equal to ten'* to describe the situation. You can switch between the circling of the	 • *'How many groups of two are there? How can we represent this with a multiplication equation?'*  $5 \times 2 = 10$ • *'There are five groups of two.'* • *'Five groups of two is equal to ten.'* • *'Five times two is equal to ten.'*

	two different group sizes several times emphasising that the array and the equation show both five groups of two *and* two groups of five. Finally, compare the arrays with the different group sizes highlighted, encouraging children to describe what's the same and what's different. Draw attention to the fact that the product remains the same.	• *'How many groups of five are there?'*  $5 \times 2 = 10$ • *'There are two groups of five.'* • *'Five, two times is equal to ten.'* • *'Five times two is equal to ten.'* • *'What's the same?'* • *'What's different?'* 

1:2	Repeat the previous step, now starting by looking at the groups of five and the equation $2 \times 5 = 10$. Then focus on the groups of two and how this can still be represented by the same equation.	• *'How many groups of five are there?'* • *'How can we represent this with a multiplication equation?'*  $2 \times 5 = 10$ • *'There are two groups of five.'* • *'Two groups of five is equal to ten.'* • *'Two times five is equal to ten.'* • *'How many groups of two are there?'*  $2 \times 5 = 10$ • *'There are five groups of two.'* • *'Two, five times is equal to ten.'* • *'Two times five is equal to ten.'*

<table>
<tr>
<td>1:3</td>
<td>Repeat steps 1:1 and 1:2 for some other multiples of two, five and ten until children are confident with seeing each array representation in two different ways, and the generalisation emerges. Use the following stem sentence (algebraic terms for teachers only): '__a__ times __b__ can represent __a__ groups of __b__. It can also represent __b__ groups of __a__ (or __a__, __b__ times).'

For example, 'Six times two can represent six groups of two. It can also represent two groups of six (or six, two times).'

Throughout steps 1:3–1:9, make multiplication charts for the two, five and ten times tables available to children (written with factors in both orders). This will allow children to focus on the structures being explored, rather than depending on fluency in the multiplication facts at this early stage.</td>
<td>Example 1:
<table>
<tr><td></td><td></td></tr>
<tr><td colspan="2">$6 \times 2 = 12$</td></tr>
<tr><td>'Six times two can represent six groups of two.'</td><td>'It can also represent two groups of six (or six, two times).'</td></tr>
</table>
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<tr>
<td></td>
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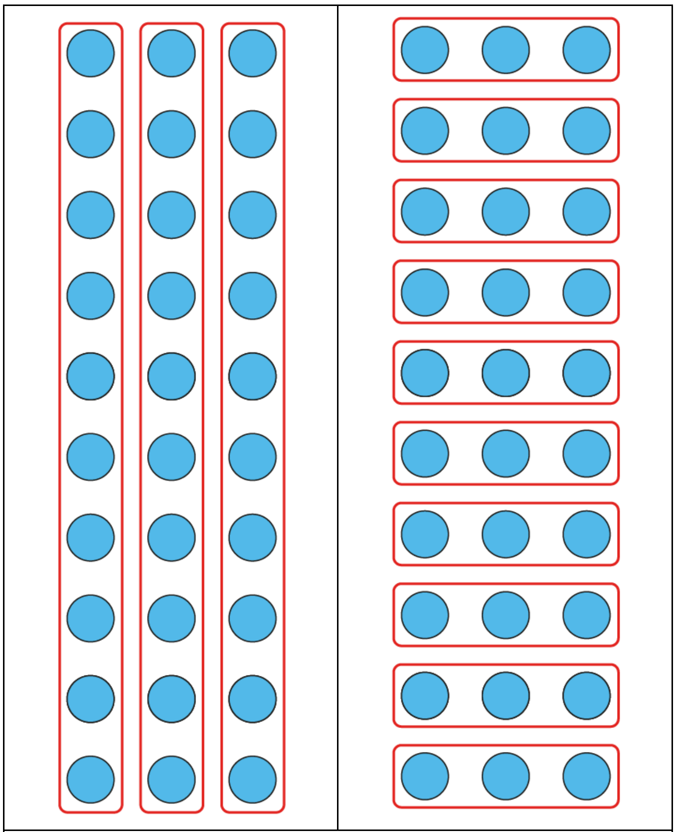
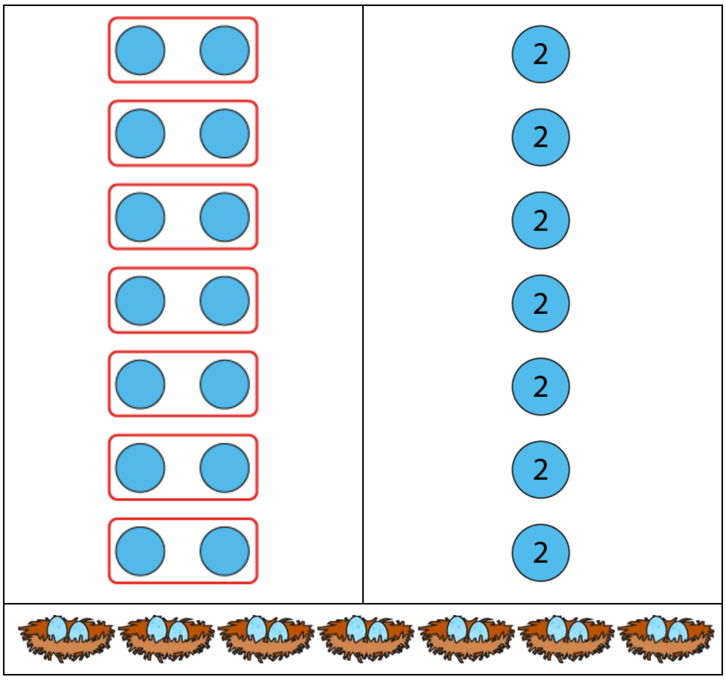
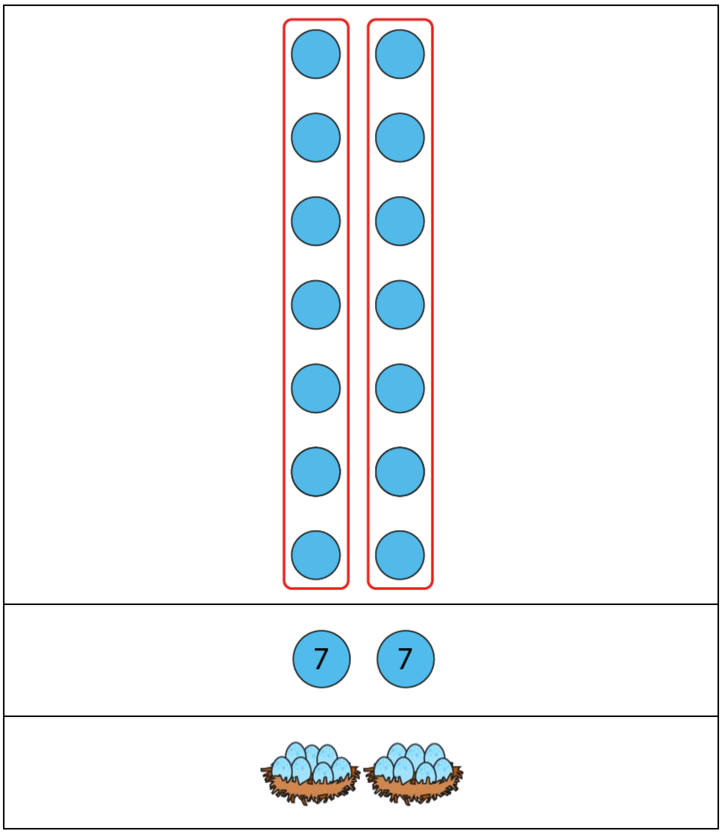
<td>Example 2:

$3 \times 10 = 30$
'Three times ten can represent three groups of ten.'
'It can also represent ten groups of three (or three, ten times).'</td>
</tr>
<tr>
<td>1:4</td>
<td>Apply the learning from steps 1:1–1:3 to some contextual examples, both where the cardinality can be seen (as with the eggs on the next page), and where the cardinality cannot be seen (as with the coins on page nine; children should already be familiar with finding the value of sets of identical coins from segment 2:1 Counting, unitising and coins). Use a variety of examples with:
groups of two or two groups of something
groups of five or five groups of something
groups of ten or ten groups of something.</td>
<td>Example 1 – cardinality visible:
'There are some eggs in some nests. What situations can the equation represent?'
$7 \times 2 = 14$</td>
</tr>
</table>

You can continue to link to arrays for support. After looking at each array, stack all of the counters in a group together and then replace the stack with a counter marked with the group size, to link to ideas of unitising, where the unit is equal to the group size.

- *'Seven times two can represent seven groups of two.'*



- *'Seven times two can also represent two groups of seven (or seven, two times).'*

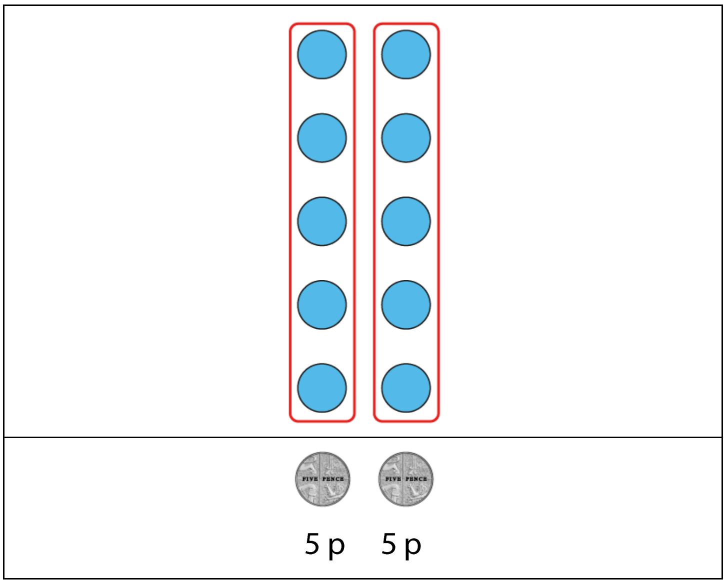


Example 2 – cardinality *not* visible:

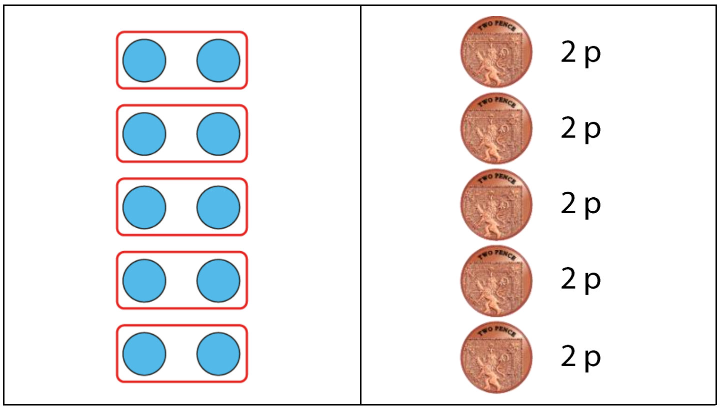
'I have some coins. What situations can the equation represent?'

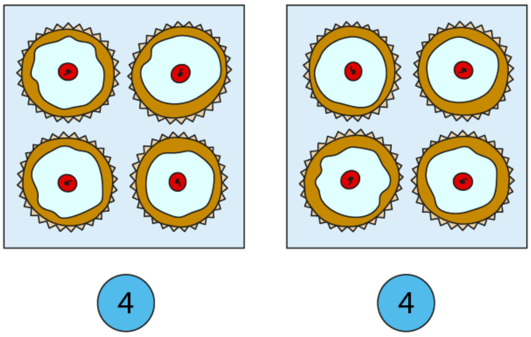
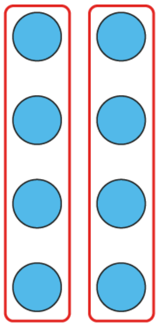
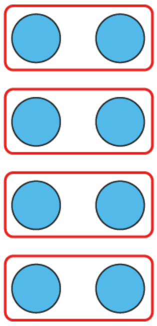
$2 \times 5 = 10$

- *'Two times five can represent two groups of five.'*



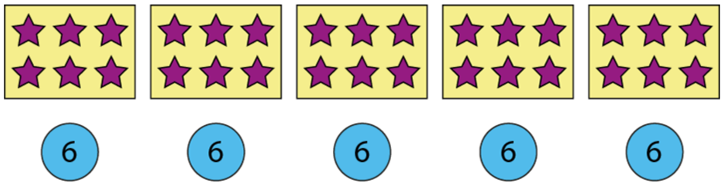
- *'Two times five can also represent five groups of two (or two, five times).'*

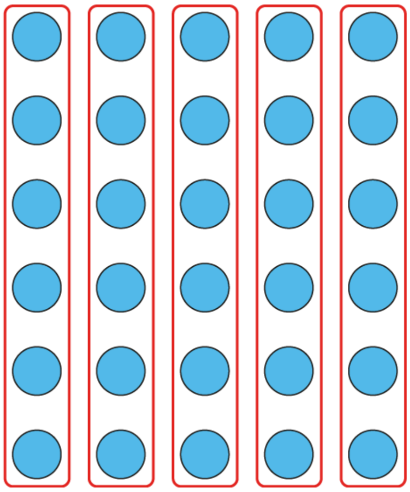


1:5	Now draw attention to the fact that we can use facts from the two times table to solve contextual problems about two equal groups. Work through several examples until children are confident in applying two times table facts to problems about two groups. Initially, use counters and arrays, as shown opposite. Then gradually remove this scaffolding as children become more comfortable with the structure and are able to use just the language of multiplication and interpretation of the equations to solve the problems; for example, immediately recognising that *2 × 4* is also a representation of two, four times, which they know is equal to eight. As you solve each problem, make a point of referring to the two times table chart each time, working towards the generalisation: ***'If there are two equal groups, we can use the two times table.'***	*'There are two boxes of cakes. Each box contains four cakes. How many cakes are there altogether?'*   $2 \times 4 = \square$ • *'Two groups of four is equal to two, four times.'*  • *'I know that two, four times is equal to eight, so I know that two groups of four is equal to eight.'* • *'Two fours are eight.'* $2 \times 4 = 8$ • *'There are eight cakes altogether.'*

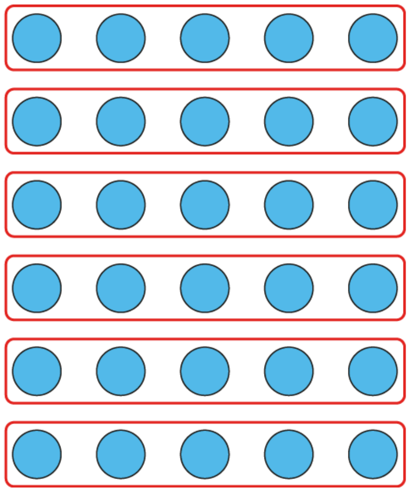
<table>
<tr>
<td>1:6</td>
<td>Repeat step 1:5, now looking at how facts from the five times table can be used to solve problems about five equal groups. Again, work through several examples until children are confident.

Use the generalisation: 'If there are five equal groups, we can use the five times table.'</td>
<td>'There are five sheets of stickers. Each sheet contains six stickers. How many stickers are there altogether?'




$5 \times 6 = \square$

• 'Five groups of six is equal to five, six times.'

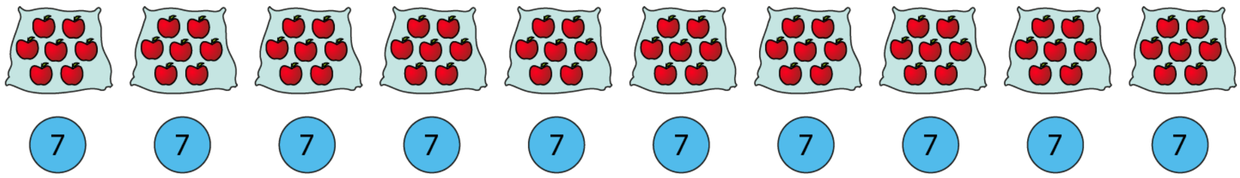


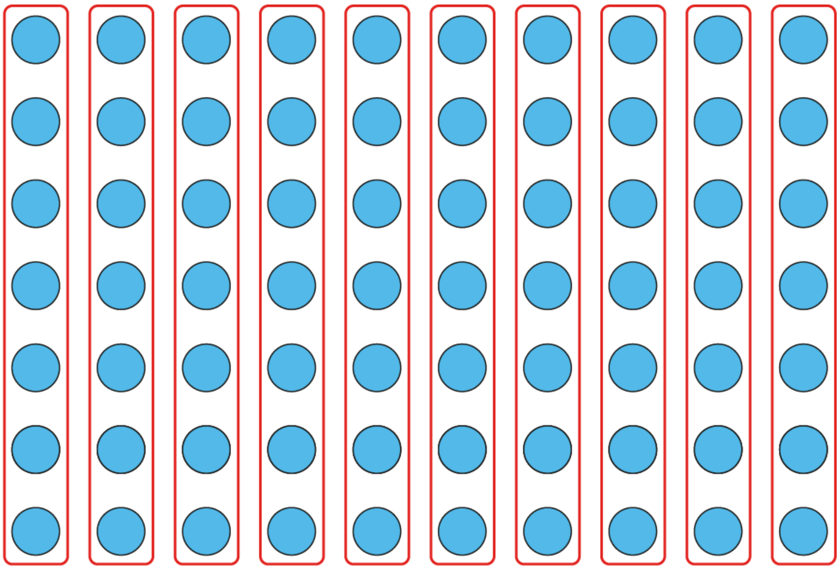
• 'I know that five, six times is equal to thirty, so I know that five groups of six is equal to thirty.'
• 'Five sixes are thirty.'
$5 \times 6 = 30$
• 'There are thirty stickers altogether.'</td>
</tr>
</table>

1:7 Repeat again, now looking at how facts from the *ten* times table can be used to solve problems about *ten* equal groups.

Use the generalisation: '***If there are ten equal groups, we can use the ten times table.'***

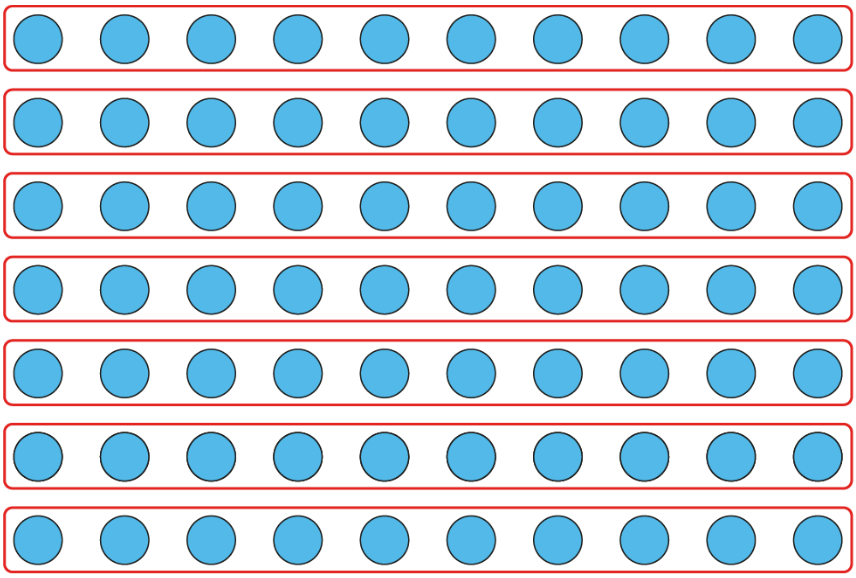
'There are ten bags of seven apples. How many apples are there altogether?'





$10 \times 7 = \square$

- *'Ten groups of seven is equal to ten, seven times.'*



- *'I know that ten, seven times is equal to seventy, so I know that ten groups of seven is equal to seventy.'*

- *'Ten sevens are seventy.'*

  $10 \times 7 = 70$

- *'There are seventy apples altogether.'*

1:8

Bring together the learning from steps *1:6–1:8* by looking at a range of contextual problems and identifying which times table can be used to help solve each of them. Use the stem sentence: ***'If there are ___ equal groups, we can use the ___ times table.'***

Then sort a range of abstract problems according to which times table can be used. In this case the problems don't have a grouping interpretation; instead, use the more general stem sentence: ***'If ___ is a factor, we can use the ___ times table.'***

To promote and assess depth of understanding you can use a dòng nǎo jīn problem, such as:

- *'Mrs Morgan has two bags. Each bag contains five loaves of bread.'*
  - *'Angie says that she can use the five times table to solve this.'*
  - *'Ben says he can use the two times table to solve this.'*

  'Who is correct? Why?'

Contextual problems:

'Fill in the missing numbers in the table to show what times table you can use to solve each problem.'

Problem	Known times table I can use (2, 5 or 10)
'There are five children. Each child has four conkers. How many conkers are there altogether?'	
'There are two teams. Each team has twelve bean bags. How many bean bags are there altogether?'	
'Five children each have three balloons. How many balloons is this altogether?'	
'Ten children each have three meatballs on their plate. How many meatballs are there altogether?'	

Abstract problems:

'Sort the calculations according to which times table can be used to solve them.'

5×4 $\quad 2 \times 9$ $\quad 7 \times 5$ $\quad 10 \times 12$ $\quad 6 \times 2$

10×8 $\quad 12 \times 5$ $\quad 2 \times 7$ $\quad 3 \times 10$ $\quad 5 \times 11$

Can use 2 times table	Can use 5 times table	Can use 10 times table

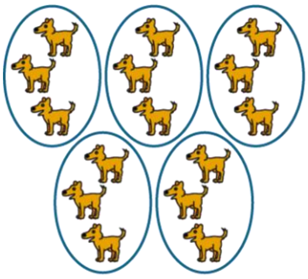
1:9

At this point, provide children with practice, including:

- writing one equation to represent two given grouping contexts
- interpreting a given equation with two grouping contexts
- pairs of oral questions, for example:
  - *'Three groups of two is equal to…?'*
  - *'Two groups of three is equal to…?'*
  - *'Four fives is equal to…?'*
  - *'Five fours is equal to…?'*
- missing-number/symbol problems (for the missing-symbol problems opposite, encourage children to draw an array for support)
- word problems, for example:
  - *'If one ice cream costs £2, how much do four ice creams cost?'*
  - *'If one book costs £4, how much do two books cost?'*
  - *'If nine children each have 5 p, how much do they have altogether?'*
  - *'If five children each have 9 p, how much do they have altogether?'*
  - *'There are eight buckets, each containing ten litres of water. How many litres of water are there altogether?'*
  - *'A dressmaker has ten pieces of ribbon. Each piece is eight centimetres long. How much ribbon is this in total?'*
  - *'There are seven players on a netball team. When two teams are playing, how many players are there altogether?'*
  - *'Five children each have eleven conkers. How many conkers do they have altogether?'*
  - *'A bag of potatoes weighs three kilograms. There are ten bags of*

Writing an equation to represent two contexts:

'Write one equation to represent both pictures.'



A

B

Interpreting an equation with two contexts:

- *'Draw an array to match the equation.'*

  $3 \times 2 = 6$
- *'Complete the two pictures to match the equation.'*

Missing-number/symbol problems:

'Fill in the missing numbers or symbols (<, > or =).'

$2 \times 5 = 5 \times \square$

$7 \times \square = 10 \times \square$

$0 = 2 \times \square$

$8 \times \square = 40$

$0 = \square \times 2$

$5 \times \square = 40$

$2 \times 4 \bigcirc 4 \times 2$

$5 \times 2 \bigcirc 2 \times 5$

$4 \times 2 \bigcirc 2 \times 5$

potatoes. How much do the potatoes weigh altogether?'

Initially, scaffold the word problems by providing a multiplication equation with missing product, then gradually reduce the scaffolding until children must interpret the problems themselves. Also, note that the first few problems are provided in pairs to support children, while the later problems require children to work straight away with two, five or ten equal groups.

Until children achieve fluency in the two, five and ten times table facts, keep the multiplication charts available for reference.

For the dòng nǎo jīn problem about representing the mittens, children may argue that there are seven ones or one seven; this is of course a correct interpretation of the mittens as a whole, and this should be acknowledged; however, discuss which equations represent the grouping shown in the picture, namely three groups of two and one more. Support children to see that this can be written both as $3 \times 2 + 1$ and $2 \times 3 + 1$ by using familiar language patterns (*'three groups of two, plus one'* or *'two, three times, plus one'*).

Dòng nǎo jīn:

- *'Fill in the missing numbers.'*

  $\square \times 5 = 5 \times \square$

  'Can you find another answer?'
  'And another?'

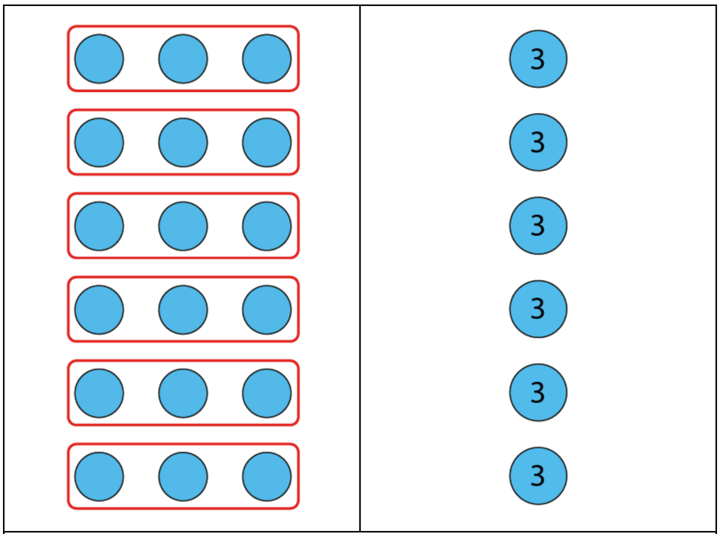
- *'Which descriptions and expressions represent the mittens?'*

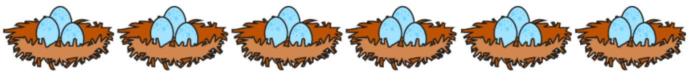
	***Do* represent the mittens (✓) or *Do not* represent the mittens (✗)**
7 groups of 1	
1 group of 7	
2 groups of 3 and 1 more	
3 groups of 2 and 1 more	
7×1	
1×7	
$3 \times 2 + 1$	
$2 \times 3 + 1$	

1:10	Extend the idea of commutativity to situations where two, five or ten are *not* one of the factors. Children already have experience of representing equal groups of various sizes with multiplication expressions (segment *2.2 Structures: multiplication representing equal groups*). Now, include the product (i.e. use *equations* not *expressions*) so that it is clear that when the factors are swapped, the product remains the same; however, always *provide* the product rather than expecting children to *calculate* it, so that the focus is on the structure, rather than calculation of the 'answer'. Throughout, focus on what each factor represents.
	
	First, explore the idea by comparing equations and by representing a particular grouping context (for example, four groups of three) with two different multiplication equations (e.g., $4 \times 3 = 12$ and $3 \times 4 = 12$). This builds on segment *2.3 Times tables: groups of 2 and commutativity (part 1), Teaching point 3*.
		Comparing equations:
		$4 \times 3 = 12$
		$3 \times 4 = 12$
		• *'What's the same?'*
		  • *'In both equations "3" and "4" are factors and "12" is the product.'*
		• *'What's different?'*
		  • *'The factors are written in different orders.'*
		
		One interpretation, two equations:
		
		$3 + 3 + 3 + 3 = 12$
		• *'There are four groups of three scoops of ice cream.'*
		$4 \times 3 = 12$
		• *'There are three scoops of ice cream, four times.'*
		$3 \times 4 = 12$
		• *'The "4" represents the number of ice-cream cones.'*
		• *'The "3" represents the number of scoops of ice cream in each cone.'*

1:11	Now look at how a particular multiplication equation can have two different grouping interpretations, using contextual examples accompanied by array representations (in the same way as in *step 1:4*); for now, use examples where the cardinality is visible. When working with arrays, as you get to larger numbers, you may wish to include the row and column factor headers (as shown on the array chart in segment *2.3 Times tables: groups of 2 and commutativity (part 1)*, step *2:7*) to make it easier for children to quickly see the size of the array.	*'There are some eggs in some nests. What situations can the equation represent?'* $6 \times 3 = 18$

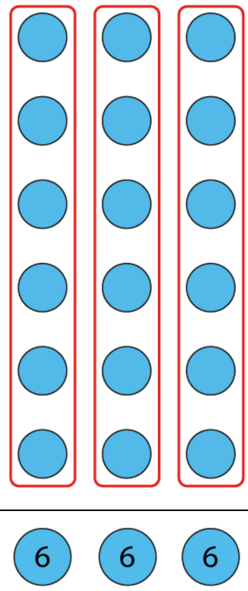
- *'Six times three can represent six groups of three.'*

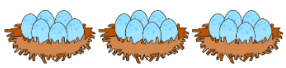




- *'The "6" represents the number of nests.'*
- *'The "3" represents the number of eggs in each nest.'*

- *'Six times three can also represent three groups of six (or six, three times).'*





- *'The "6" represents the number of eggs in each nest.'*
- *'The "3" represents the number of nests.'*

1:12

Similar to step *1.9*, provide children with practice related to commutativity, with factors other than two, five or ten, including:

- writing a pair of *expressions* (not *equations*, as children are not expected to calculate the product) to match a given grouping representation (one interpretation, two equations)
- interpreting a given equation with two grouping contexts (one equation, two interpretations)
- missing-number/symbol problems (for the missing-symbol problems opposite, encourage children to draw an array for support)
- using given multiplication facts to find related facts.

Missing-number/symbol problems:

'Fill in the missing numbers or symbols (<, > or =).'

$7 \times 4 = \square \times 7$ $4 \times 9 = 9 \times \square$

$3 \times 7 = 21$ $40 = 10 \times 4$

$7 \times 3 = \square$ $40 = \square \times 10$

$5 \times 4 \bigcirc 4 \times 5$

$5 \times 4 \bigcirc 4 \times 6$

$6 \times 4 \bigcirc 4 \times 6$

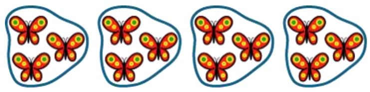
Dòng nǎo jīn:

- 'Fill in the missing numbers.'

  $\square \times \square = \square \times \square$

  'Can you find another answer?'
  'And another?'

- Some children look at this picture and write expressions.

  Sally: 3 × 4

  Ahmed: 4 × 3

  Kumiko: 4 + 4 + 4

  'Who has represented the picture correctly?'
  (Sally has represented the picture correctly as three, four times. Ahmed has represented the picture correctly as four groups of three. Kumiko's expression, whilst giving the correct total number of butterflies, does not represent the grouping shown in the picture. You can ask children to explain what each number represents to help them reason about the expressions.)

1:13

To complete this teaching point, ensure that children understand the two possible grouping interpretations when '1' is a factor.

In segment *2.4 Times tables: groups of 10 and of 5, and factors of 0 and 1*, children learnt the generalisation: ***'When one is a factor, the product is equal to the other factor.'*** At that stage the exploration focused on pattern spotting with the times tables. Now explore this idea with the two interpretations (with '1' as either the number of groups or the group size).

Because multiplication is often seen as 'making bigger', children can find this a difficult point. Use familiar representations, such as Numberblocks or coins, alongside arrays and counters, as exemplified opposite and on the next page.

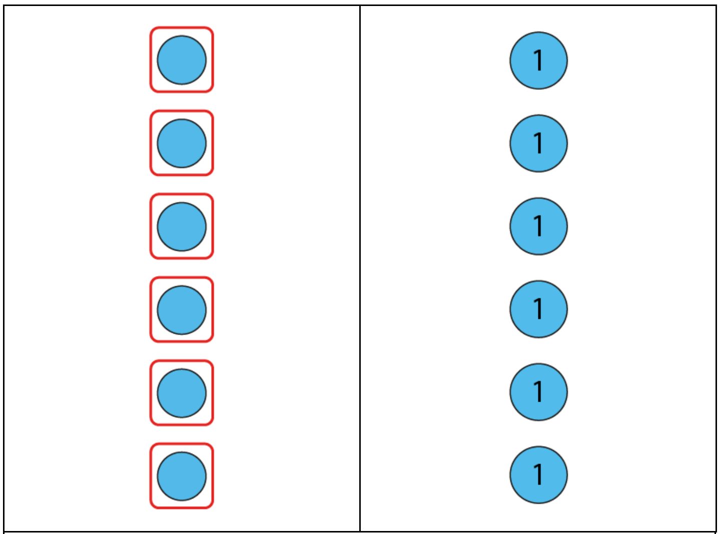
Use the following dòng nǎo jīn question to assess children's depth of understanding:

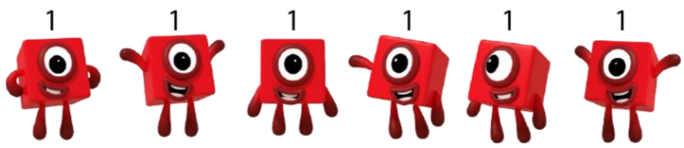
- *Jen says:* 'I have ten coins.'
- *Bo says:* 'I have one coin.'
- *Meg says:* 'Jen must have more money because she has more coins.'
- *Jordan says:* 'Bo must have less money because he has only one coin.'
- *Libby says:* 'They could both have the same amount of money.'
- *'Who is right? Explain.'*

Numberblocks:

$6 \times 1 = 6$

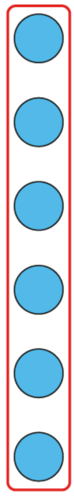
- *'Six times one can represent six groups of one.'*



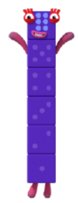


- *'The "6" represents the number of characters.'*
- *'The "1" represents the value of the characters.'*

- *'Six times one can also represent one group of six (or six, one time).'*



6

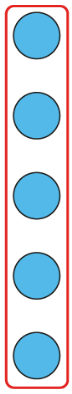


- *'The "6" represents the value of the character.'*
- *'The "1" represents the number of characters.'*

Coins:

$1 \times 5 = 5$

- *'One times five can represent one group of five.'*



5 p

- *'The "1" represents the number of coins.'*
- *'The "5" represents the value of the coin.'*

- *'One times five can also represent five groups of one (or one, five times).'*

1 p

1 p

1 p

1 p

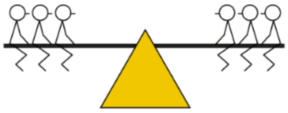
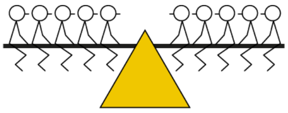
1 p

- *'The "1" represents the value of each coin.'*
- *'The "5" represents the number of coins.'*

Teaching point 2:

If two is a factor, knowledge of doubling facts can be used to find the product; problems about doubling can be solved using facts from the two times table.

Steps in learning

	Guidance	Representations
2:1	This and the next teaching point build on learning from *Spine 1: Number, Addition and Subtraction*, segment *1.7*, where doubling and halving is discussed in terms of additive reasoning, i.e.: • *'When both addends are the same, we are doubling.'* E.g., $4 + 4 = 8$ *'Double four is equal to eight.'* • *'Halving is the inverse of doubling.'* E.g., $8 - 4 = 4$ *'Half of eight is equal to four.'* This teaching point also builds on *Teaching point 1* above, since doubling is initially discussed in terms of two equal groups. The following key points are explored: • Doubling is structurally equivalent to two equal groups (steps *2:1–2:3*). • Known two times table facts can be used to solve problems about 'doubling' or 'twice as many/much' (step *2:4*). • Doubling facts and strategies are extended to a larger range of numbers (steps *2:5–2:6*). • Known doubling facts can be used to solve problems about two equal groups (step *2:7*). • Doubling facts can be used to solve problems where two is a factor (problems about two equal groups, about groups of two, and abstract problems with no contextual grouping interpretation) (step *2:8*). Begin by looking at balancing a seesaw, as in *Spine 1*, segment *1.7*; this	$2 \times 3 = 6$ $3 \times 2 = 6$  $3 + 3 = 6$ • *'There are two groups of three.'* • *'There are three, two times.'* • *'This is the same as double three.'* $2 \times 5 = 10$ $5 \times 2 = 10$  $5 + 5 = 10$ • *'There are two groups of five.'* • *'There are five, two times.'* • *'This is the same as double five.'*

representation helps us to see, in additive reasoning, that we have two equal addends and, in multiplicative reasoning, that we have two equal groups. Start with more than one stick-child on each side of the seesaw (for example, three on each side), asking:

- *'How many groups of stick-children are there on the seesaw?'*
- *'How many stick-children are there in each group?'*
- *'How many stick-children are there altogether?'*

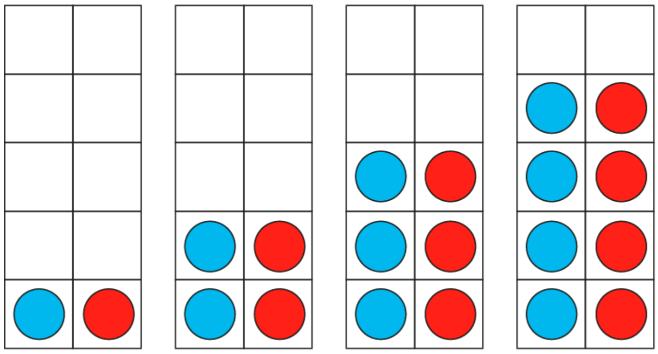
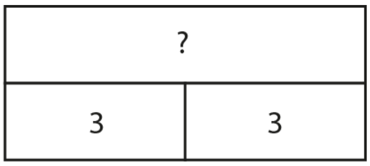
Then ask children to write the two possible multiplication equations (e.g., $2 \times 3 = 6$ and $3 \times 2 = 6$) and describe the situation using the following familiar language:

- *'There are two groups of three.'*
- *'There are three, two times.'*

Then write the addition equation (e.g., $3 + 3 = 6$), and describe the situation using doubling language from *Spine 1*, for example: '*Double three is six.*'

Work through the same process with a different number of stick-children on the seesaw, encouraging children to describe each case with both grouping language and doubling language:

- ***'There are two groups of ___.'***
- ***'There are ___, two times.'***
- ***'This is the same as double ___.'***

2:2	Now use the generalised representation from *Spine 1: Number, Addition and Subtraction*, segment *1.7* – tens frames and counters. Work through systematically from double one upwards, writing both multiplication equations, and the addition equation, in each case. Use the following simplified stem sentences, with emphasis on the equivalence between 'two times' and 'double' to describe each tens frame: ***'___, two times is the same as double ___.'*** Then look at the complete set of tens frames, asking: • *'What's the same?'* ('2' is always a factor; in each frame there is the same number of blue and red counters; the total number of counters is always even.) • *'What do you notice about the multiplication equations?'* (They are the same equations as those in the two times table.)	$2 \times 1 = 2$ $2 \times 2 = 4$ $2 \times 3 = 6$ $2 \times 4 = 8$ $1 \times 2 = 2$ $2 \times 2 = 4$ $3 \times 2 = 6$ $4 \times 2 = 8$ …  $1 + 1 = 2$ $2 + 2 = 4$ $3 + 3 = 6$ $4 + 4 = 8$ … • *'One, two times is the same as double one.'* • *'Two, two times is the same as double two.'* • *'Three, two times is the same as double three.'* • *'Four, two times is the same as double four.'*
2:3	Use bar models to explore doubling further. In general, it is not important that a value is given to the parts or the whole; the focus should be on the key structural features: • There are two parts. • The parts are equal. • The whole is double one of the parts. You can draw this out using the following questions: • *'How many parts are there?'* (two) • *'What can you tell me about the parts?'* (they are equal/the same) • *'How many parts equal the whole?'* (the two parts equal the whole) • *'What is the relationship between the whole and one of the parts?'* (the whole is double one of the parts).	 $2 \times 3 = \square$ • *'Two groups of three.'* • *'Three, two times.'* • *'Double three.'*

2:4

Now that children understand that multiplication with two equal groups is equivalent to doubling, explore how problems about doubling can be solved using facts from the two times table.

Present a contextual problem, using the language of doubling, for example: '*This morning there were three birds on the wall. Now there are double that number. How many birds are on the wall now?*'

Ask children to:

- represent the problem with a bar model
- write a multiplication expression with two as a factor
- describe the doubling problem in terms of two equal groups
- use the two times table fact to find the product.

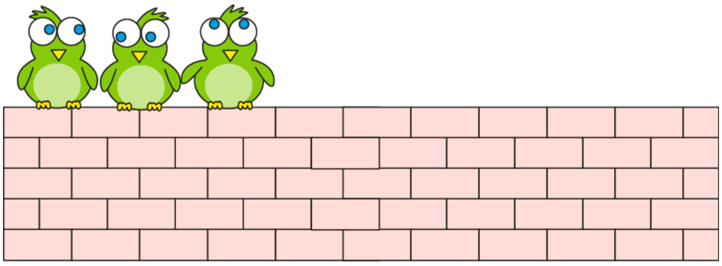
Then repeat for another contextual problem with the language of '*twice as many/much*' (to demonstrate that this is equivalent to '*double*'). In both cases, draw attention to the fact that we are being given the value of one group or one of the parts, and need to find the value of two groups (double/twice) or two parts.

Note that problems phrased as *'twice as far'*, '*twice as long*', '*twice as heavy*', and so on, have a scaling structure. This will be explored fully in segment *2.17 Structures: multiplication and division as scaling*.

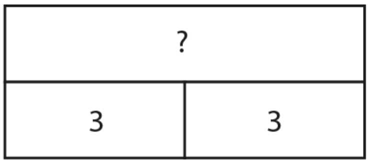
Work towards the following generalisation: '***If we need to double/find twice the amount, we can use facts from the two times table.'***

Contextual problem – 'double':

'This morning there were three birds on the wall. Now there are double that number. How many birds are on the wall now?'



- *'Double three is the same as two groups of three.'*



2×3

- *'I know that two times three is equal to six.'*

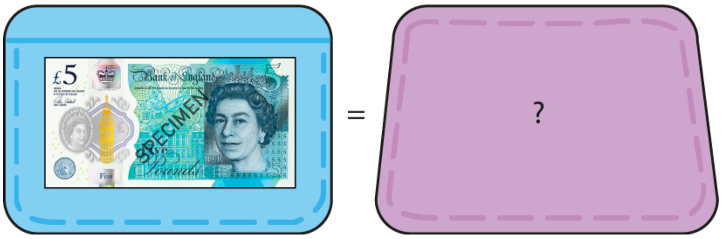
$2 \times 3 = 6$

- *'So, double three is equal to six.'*
- *'There are six birds on the wall now.'*

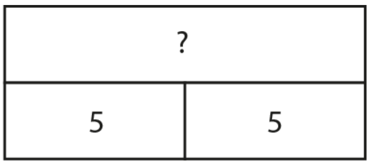
Contextual problem – 'twice as much':

'Bobby has £5. Kali has twice as much as Bobby. How much does Kali have?'

Bobby Kali



- *'Twice as much means double.'*
- *'Double five is the same as two groups of five.'*



2×5

- *'I know that two times five is equal to ten.'*

$2 \times 5 = 10$

- *'So, double five is equal to ten.'*
- *'Kali has ten pounds.'*

2:5

Before using doubling to solve multiplication problems with a factor of two (steps *2:7* and *2:8*), spend some time reviewing and extending children's doubling facts and strategies.

In *Spine 1: Number, Addition and Subtraction*, segment *1.7*, children learnt doubling facts for numbers up to and including five (i.e. from double one to double five). In *Spine 1*, segment *1.10*, children learnt doubling facts from double six to double ten; they explored how six is equal to five plus one, so double six is to double five plus double one.

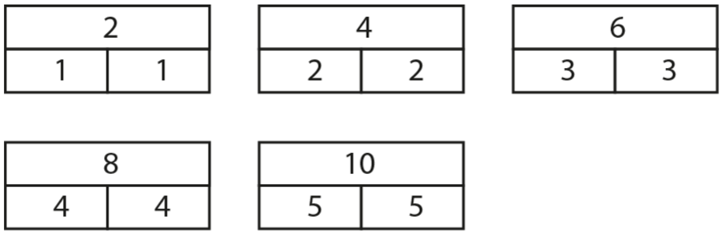
Review doubling facts for one to five, then use similar representations to *Spine 1*, segment *1.10* (counters and tens frames, or base-ten number boards) to work through doubling six to ten, using the 'five-and-a-bit' structure of these numbers. Double nine can also be explored as double ten minus double one.

A practical way to expose how the 'five-and-a-bit' structure is useful in doubling, is for children to work in pairs and use their hands; for example, for double six, each child holds up six fingers (a five and a one), then the two children place their hands together, palms facing: the two fives come together to make ten and the two ones come together to make two.

Note that this strategy is based on the distributive law of multiplication, e.g., $2 \times 7 = (2 \times 5) + (2 \times 2)$, which children will explore formally in segment *2.10 Connecting multiplication and division, and the distributive law*; for now, use just the representations and language for children to reason with, not the mixed-operation equations.

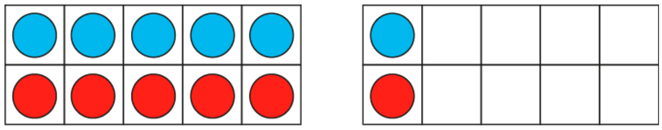
Remind children of the following generalisation from *Spine 1*, segment

Known doubling facts for one to five:

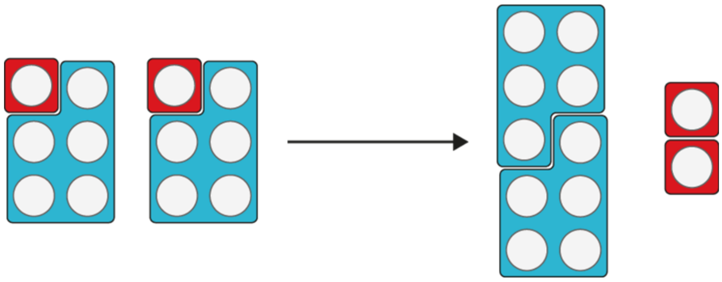


Doubling strategy for six to ten – example 1:

- Tens frames and counters:



- Base-ten number boards:



'Six is five plus one, so double six is double five plus double one.'

$5 + 5 = 10$

$1 + 1 = 2$

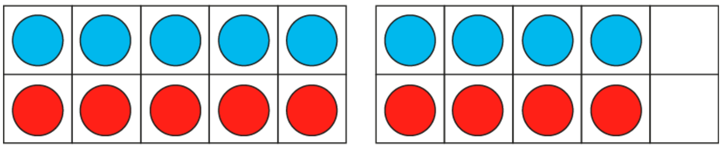
$10 + 2 = 12$

so

$6 + 6 = 12$

1.10, and encourage them to use this when checking their doubling answers: ***'Doubling a whole number always gives an even number.'***

Doubling strategy for six to ten – example 2:

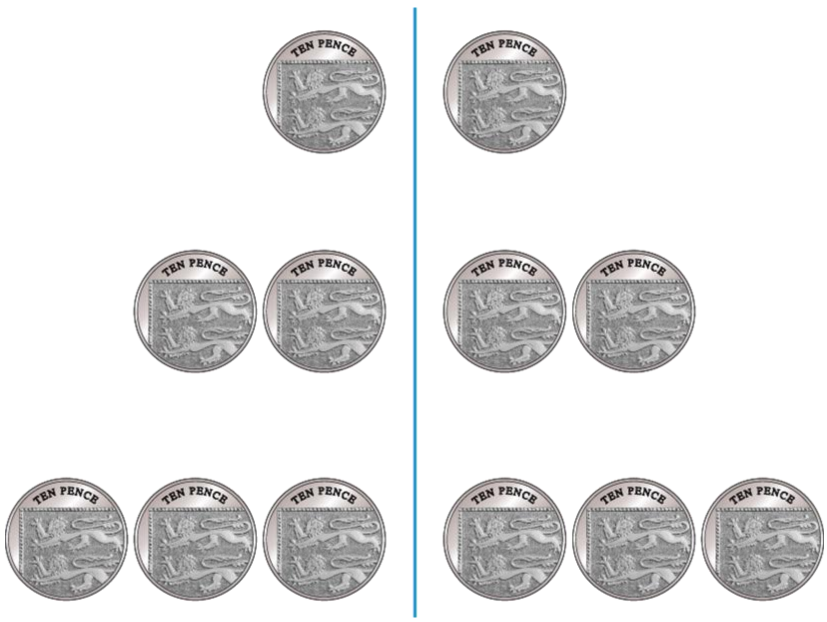


'Nine is five plus four, so double nine is double five plus double four.'	*'Nine is ten minus one, so double nine is double ten minus double one.'*
5 + 5 = 10	10 + 10 = 20
4 + 4 = 8	1 + 1 = 2
10 + 8 = 18	20 – 2 = 18
so	so
9 + 9 = 18	9 + 9 = 18

2:6 Now extend doubling strategies and facts to two-digit numbers, including:

- doubling multiples of ten, using the language of unitising and noting that doubling a multiple of ten always gives an *even* multiple of ten; you can use 10 p coins to support the unitising structure
- doubling the numbers 11–15 by partitioning and using known facts
- doubling two-digit numbers with a ones digit of five (e.g., 25, 35…), by partitioning into tens and ones, and noting that doubling these numbers always gives an *odd* multiple of ten; this, along with doubling multiples of ten, will be useful in *Teaching point 4*, where children explore the relationship between the five and ten times tables.

Doubling multiples of ten:



- *'Double one ten is two tens.'*
- *'Double ten is twenty.'*

10 + 10 = 20

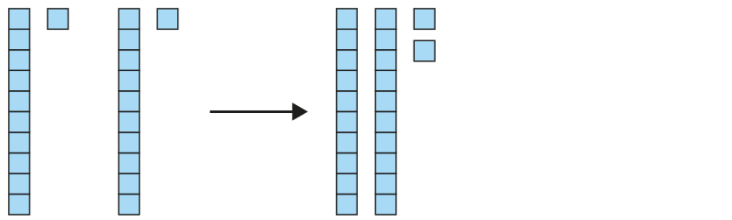
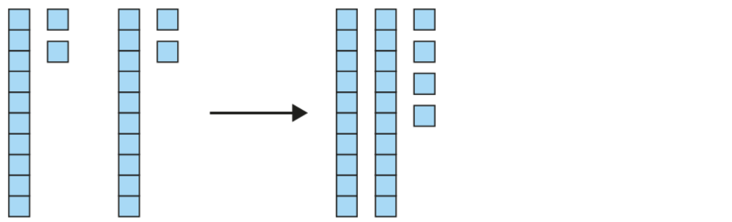
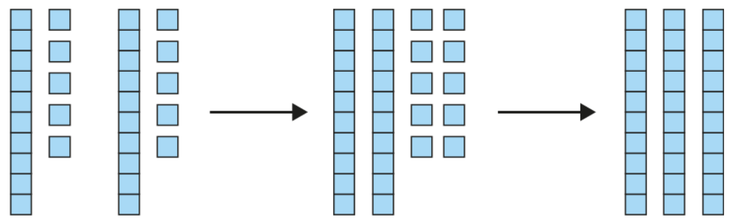
- *'Double two tens is four tens.'*
- *'Double twenty is forty.'*

20 + 20 = 40

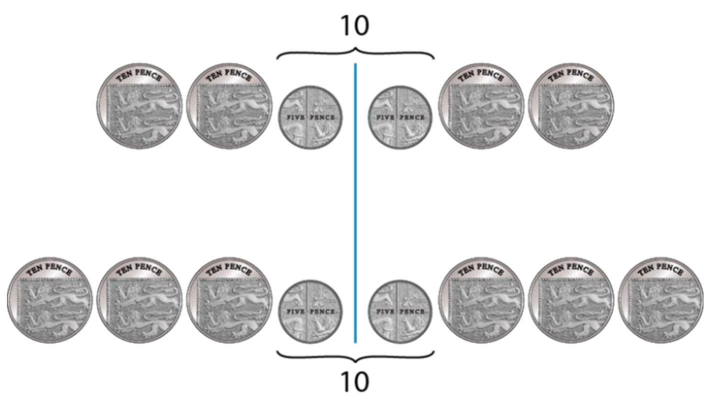
- *'Double three tens is six tens.'*
- *'Double thirty is sixty.'*

30 + 30 = 60

Doubling 11–15:

	double 11 = double 10 + double 1 = 20 + 2 = 22
	double 12 = double 10 + double 2 = 20 + 4 = 24
⋮	⋮
	double 15 = double 10 + double 5 = 20 +10 = 30

Doubling two-digit numbers with a ones-digit of five:

	double 25 = double 20 + double 5 = 40 +10 = 50
	double 35 = double 30 + double 5 = 60 +10 = 70
⋮	⋮

<table>
<tr>
<td>2:7</td>
<td>
Children should now:

understand that multiplication with two equal groups is equivalent to doubling
have a range of doubling facts and strategies at their disposal.

Bring this knowledge together now, exploring how known doubling facts and strategies can be used to solve problems about two equal groups.

Present a contextual problem, for example: ‘There are two packets of dog bones. There are six bones in each packet. How many bones are there altogether?’

Ask children to:

represent the problem with a bar model
write a multiplication expression with ‘2’ as a factor
describe the problem in terms of doubling
use a known doubling fact to solve the problem, with the following stem sentence: ‘I know double ___ is ___, so two groups of ___ is ___.’

Work through several problems, including those where the cardinality can be seen (as with the dog bones) and those where it can’t (for example, two 5 p coins, two 4 m lengths of ribbon etc.). Throughout, use the generalisation: If there are two equal groups, we can use doubling facts.’
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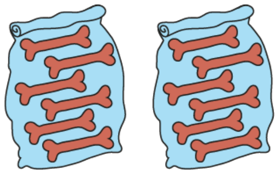
Example 1 – cardinality visible:

‘There are two packets of dog bones. There are six bones in each packet. How many bones are there altogether?’



?
6 | 6

$2 \times 6 = \square$

‘I know double six is twelve, so two groups of six is twelve.’
$2 \times 6 = 12$
‘There are twelve dog bones altogether.’

Example 2 – cardinality not visible:

‘There are two purses. Each purse contains 25 p. How much is this altogether?’

$2 \times 25 = \square$

double 25 = double 20 + double 5
= 40 + 10
= 50

‘I know that double twenty-five is fifty, so two groups of twenty-five is fifty.’
$2 \times 25 = 50$
‘There is 50 p altogether.’

</td>
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<tr>
<td>2:8</td>
<td>Now look at some abstract problems with a factor of ‘2’. Present some of the multiplication equations generated during earlier steps, written in pairs as shown opposite; ask children ‘What’s the same’, prompting them to notice that:</td>
<td>
Comparing multiplication equations with a factor of two:

$2 \times 1 = 2$ $2 \times 2 = 4$ $2 \times 3 = 6$ $2 \times 4 = 8$
$1 \times 2 = 2$ $2 \times 2 = 4$ $3 \times 2 = 6$ $4 \times 2 = 8$
</td>
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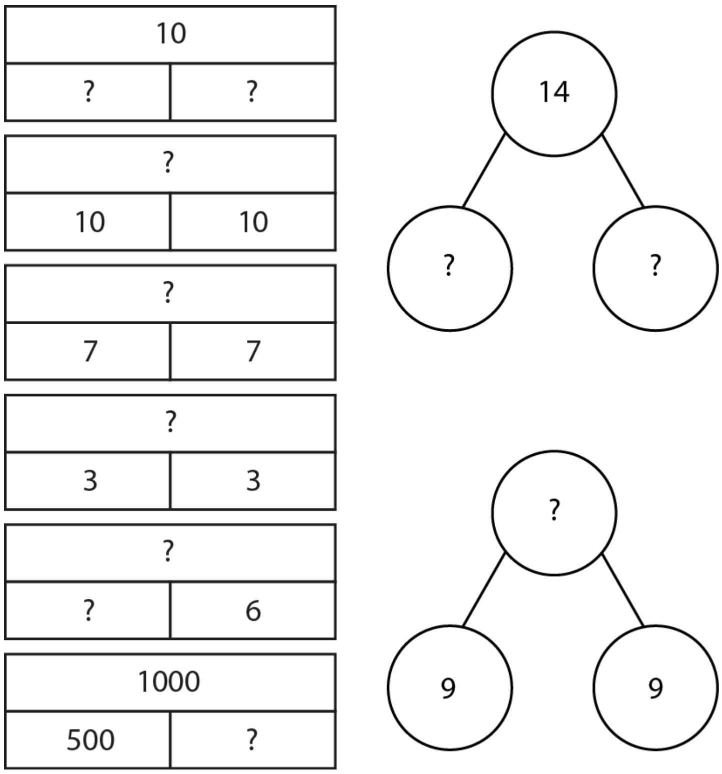
</table>

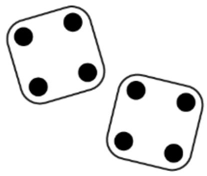
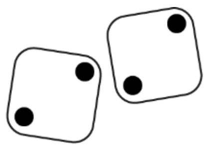
	• '2' is a factor in each equation. • The product is double the other factor. Make the following generalisation: ***'When one of the factors is two, the product is double the other factor.'*** (This takes the generalisations from earlier in the teaching point a step further, making it independent of the grouping context.) Then illustrate that this generalisation can be applied to any problem where two is a factor, even if two is the group size, rather than the number of groups.	Applying doubling facts to problems about groups of two: *'There are thirteen pairs of socks. How many socks are there altogether?'* $13 \times 2 = \square$ • *'One of the factors is two, so the product is double the other factor.'* double 13 = double 10 + double 3 = 20 + 6 = 26 • *'I know that double thirteen is twenty-six, so thirteen times two is twenty-six.'* $13 \times 2 = 26$ • *'There are 26 socks altogether.'*

2:9	Provide children with varied practice for problems about:
	• two equal groups
	• groups of two
	• 'doubling'
	• 'twice as many/much'.
	Include:
	• part–part–whole diagrams with missing numbers (note that in some of the examples opposite children will be beginning to implicitly deal with halving)
	• completing/writing multiplication equations
	• true/false style questions
	• contextual problems, for example:
	  ◦ *'Mahmud has nine marbles. Felicity has twice as many marbles as Mahmud. How many marbles does Felicity have?'*
	  ◦ *'Beth has 15 points in a computer game. If she doubles her points, what would her score be?'*
	  ◦ *'There are two packets of dog bones. There are twelve bones in each*
		Part–part–whole diagrams:
		'In all of these diagrams the parts are equal. Fill in the missing numbers.'
		

	packet. How many bones are there altogether?' • *'If there are twenty pairs of children, how many children are there altogether?'*	Completing multiplication equations: *'Fill in the missing numbers.'* $2 \times 7 = \square$ $7 \times 2 = \square$ Double 7 is equal to $\square$ 7, twice is equal to $\square$ We can write twice 6 as $\square \times 6$ double 6 = $\square$ We can write double 3 as $2 \times \square$ this is the same as $\square \times 2$ double 3 = $\square$

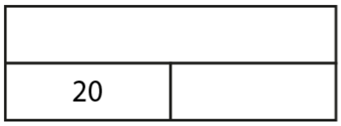
True/false style problems:

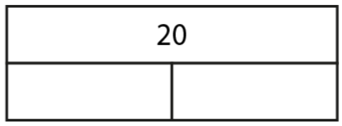
'Tick the examples that represent double four.'

	4 × 2
4 + 4	
2 × 4	

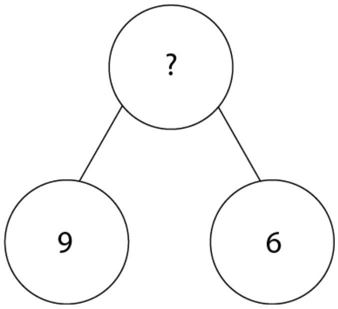
Dòng nǎo jīn:

- *'There are double the number of children outside as there are inside. There are 20 children outside. ' 'Which diagram shows this?'*





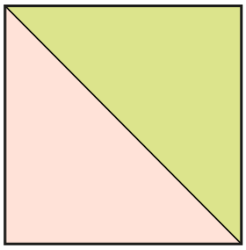
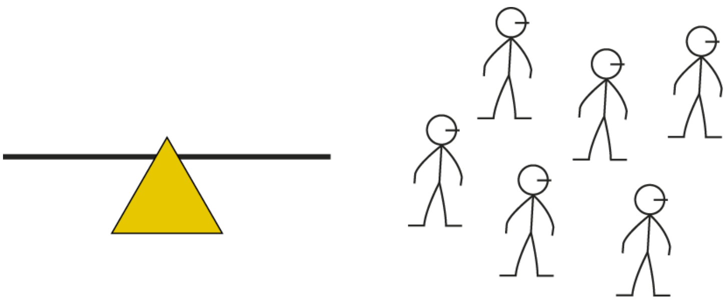
- *'Becca says that when you double a number, the result is always an even number. Is she right? Convince me. '*

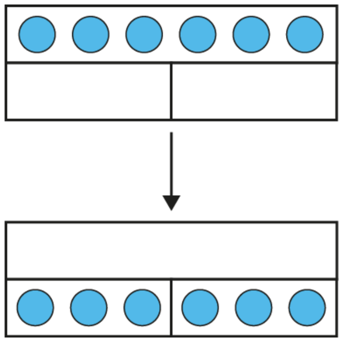
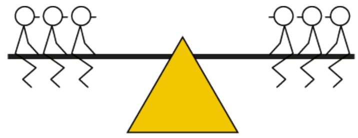
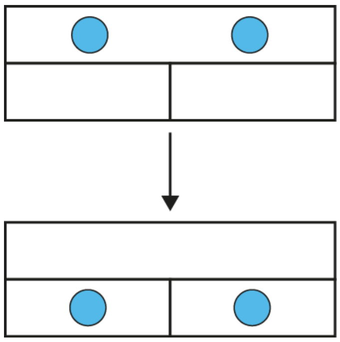
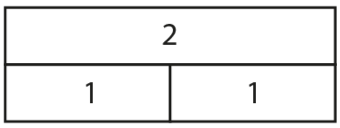
		• *'True or false? This diagram shows doubling, because there are two parts.'* 

Teaching point 3:

Halving is the inverse of doubling. Problems about halving can be solved using facts from the two times table and known doubling facts.

Steps in learning

	Guidance	**Representations**
3:1	Children already know that halving is the inverse of doubling, in the context of additive reasoning (*Spine 1: Number, Addition and Subtraction*, segment *1.7*). They will also be aware that if a whole is split into two equal parts, one of the parts is equal to one half of the whole (*Spine 3:* Fractions, segment *1.2* and opposite). Briefly remind children of these points before moving on to look at halving in the context of two equal groups and the two times table. Note that children will essentially be dividing by two in this teaching point; however, they will be doing so using multiplication or doubling facts, and will not be using the word 'division' or the associated symbol. Halving is linked to division with a divisor of two in segment *2.6 Structures: quotitive and partitive division*, *Teaching point 4*. Use the same context as in step *2:1*. Present a problem such as: '*There are six stick-children who want to play on the seesaw; half of them should sit on each side. How many is half of six?*' Ask children: • *'How many halves make a whole?'* (two) • *'So how many equal groups of children will there be?'* (two) Then connect to children's learning about multiplication and doubling by asking the following questions, and filling in the missing factors in the multiplication equations:	Half of a whole:  • *'The whole is split into two equal parts.'* • *'Each part is one half of the whole.'* Finding half of a number of objects: • *'There are six stick-children who want to play on the seesaw; half of them should sit on each side. How many is half of six?'* 

	• *'Two groups of what is equal to six?'* • *'Double what is equal to six?'* Use a printed bar model (with two equal parts) and counters alongside the multiplication equations and ask children to split the counters into two equal groups as they find the answer (children practised making equal groups in segment *2.2 Structures: multiplication representing equal groups*). Then connect back to the original problem, splitting the stick-children into two equal groups to sit on the seesaw. Describe the context using the following stem sentence: ***'There are ___ altogether; half of ___ is equal to ___.'*** Work through a few more examples, halving even numbers of stick-children who want to play on the seesaw. Draw attention to the fact that in each case, we are splitting the total number of stick-children into two equal parts.	• *'Two groups of what is equal to six?'* *'Double what is equal to six?'* $2 \times \square = 6$ $\square \times 2 = 6$  $2 \times \boxed{3} = 6$ $\boxed{3} \times 2 = 6$  • *'There are six stick-children altogether; half of six is equal to three.'* • *'Three stick-children need to sit on each side of the seesaw.'*
3:2	Continuing with the generalised representation of the bar model and counters, work systematically halving two, then four, then six, and so on. For each example: • have children use the counters to represent halving the total quantity • summarise the result on a bar model with numerals • describe the halving process using the language from step *3:1*: ***'Half of ___ is equal to ___.'*** and write the statement on the board using numerals • describe the inverse doubling process using the stem sentence: '***Double ___ is equal to ___.'*** and write the statement on the board using numerals.	  Half of 2 is equal to 1. Double 1 is equal to 2.

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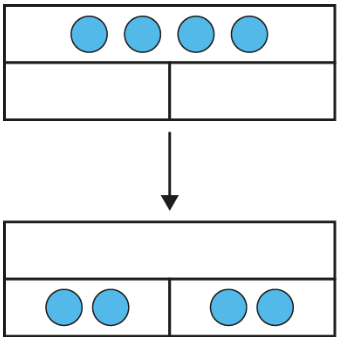
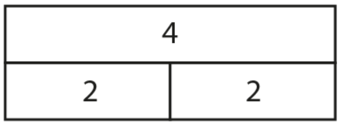
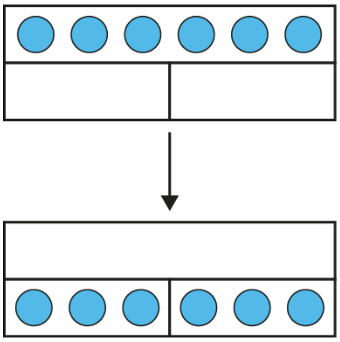
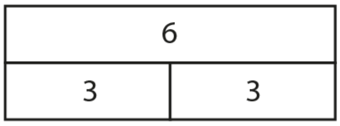



Half of 4 is equal to 2.
Double 2 is equal to 4.

Half of 6 is equal to 3.
Double 3 is equal to 6.
</td>
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<tr>
<td>3:3</td>
<td>
Once you have built up a set of examples in step 3:2, compare the bar models (with numerals) and sentences, and ask children what patterns they can spot. Draw out:

the inverse relationship between doubling and halving
that the bar models show two times table facts whether we are halving or doubling.

When children can see these patterns, add the multiplication equations alongside each bar model. Then present the next halving problem in the sequence (here, finding half of eight), and encourage children to write the next multiplication equation in the sequence to identify half. Continue to use the halving and doubling stem sentences together.
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Spotting patterns:
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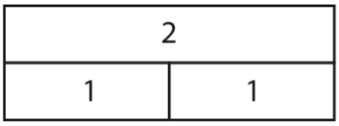
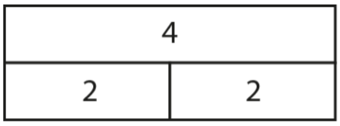
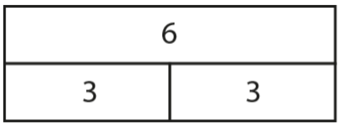
<td>

$2 \times 1 = 2$</td>
<td>Half of 2 is equal to 1.
Double 1 is equal to 2.</td>
</tr>
<tr>
<td>

$2 \times 2 = 4$</td>
<td>Half of 4 is equal to 2.
Double 2 is equal to 4.</td>
</tr>
<tr>
<td>

$2 \times 3 = 6$</td>
<td>Half of 6 is equal to 3.
Double 3 is equal to 6.</td>
</tr>
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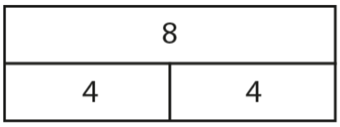
<td>

$2 \times 4 = 8$</td>
<td>Half of 8 is equal to ___.
Double ___ is equal to 8.</td>
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</table>
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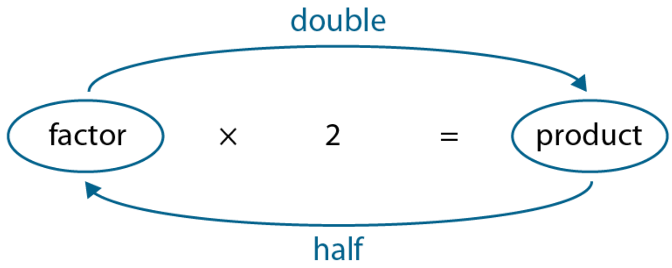
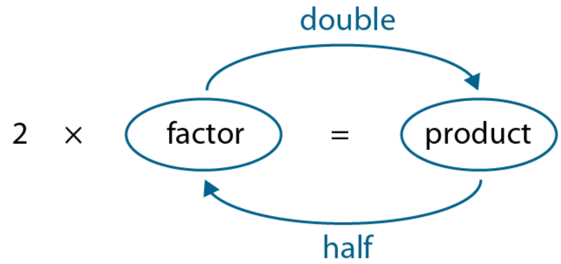
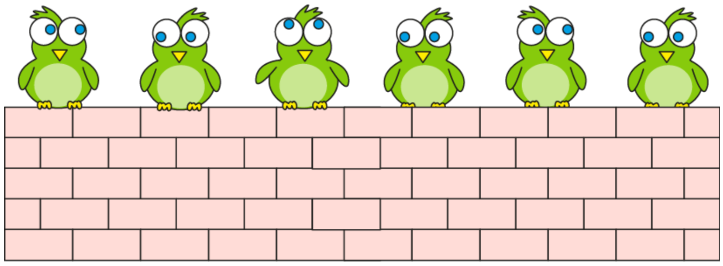
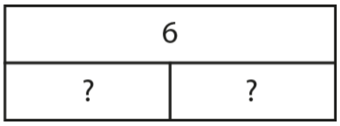
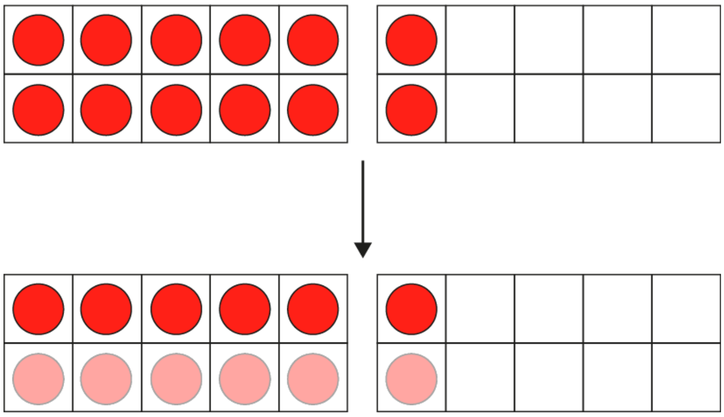
	You can summarise the relationships as shown in the final representation opposite and using the following generalisation: ***'When one of the factors is two, the other factor is half of the product.'***	Summarising the relationships:  
3:4	Now explore how problems about halving can be solved using facts from the two times table. Present a contextual problem, using the language of halving, for example: *'This morning there were six birds on the wall. Now there are half that number. How many birds are on the wall now?'* Ask children to: • represent the problem with a bar model • write a multiplication expression with a factor of '2' and with the other factor missing • use the relevant two times table fact to find the missing factor.	*'This morning there were six birds on the wall. Now there are half that number. How many birds are on the wall now?'*   $2 \times \square = 6$ $\square \times 2 = 6$ • *'I know that two times three is equal to six.'* or • *'I know that three twice is equal to six.'* $2 \times \boxed{3} = 6$ $\boxed{3} \times 2 = 6$ • *'So, half of six is equal to three.'* • *'There are three birds on the wall now.'*

<table>
<tr>
<td>3:5</td>
<td>Before using halving to solve problems about two equal groups, or groups of two, spend some time reviewing and extending children's halving facts and strategies, including the following:

Apply known doubling facts, using the stem sentence: 'I know that double ___ is ___; so half of ___ is ___.'
Halve even two-digit numbers from 12 to 28 by partitioning into tens and ones, halving the parts, and recombining (initially supported by tens frames and counters). Note that 18 could be halved using half of 20 minus half of 2.
Unitise and apply known facts to halve even multiples of ten (for example, 'Half of four is two, so half of four tens is two tens').
Halve odd multiples of ten by partitioning into ten and an even multiple of ten, applying known facts and recombining.

</td>
<td>Example 1 – applying known doubling facts:
<table>
<tr><td colspan="2">8</td></tr>
<tr><td>4</td><td>4</td></tr>
</table>
'I know that double four is equal to eight; so half of eight is equal to four.'

Example 2 – halving even teen numbers:



half of 12 = half of 10 + half of 2
= 5 + 1
= 6
'Twelve is ten plus two, so half of twelve is equal to half of ten plus half of two.'

Example 3 – halving even two-digit numbers:
half of 26 = half of 20 + half of 6
= 10 + 3
= 13
'Twenty-six is twenty plus six, so half of twenty-six is half of twenty plus half of six.'

Example 4 – halving odd multiples of ten:
half of 30 = half of 20 + half of 10
= 10 + 5
= 15
'Thirty is twenty plus ten, so half of thirty is half of twenty plus half of ten.'
</td>
</tr>
</table>

3:6

Remind children of the generalisation from step *3:3*: ***'When one of the factors is two, the other factor is half of the product.'***

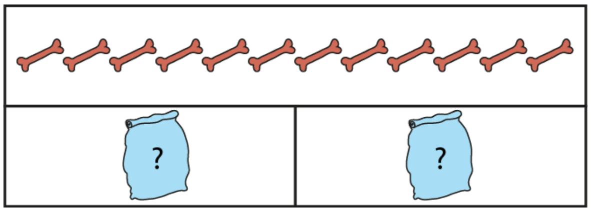
Then present a contextual problem about two groups that does *not* use the language of halving. Ask children to:

- write a multiplication equation with missing factor to represent the problem
- represent the problem with a bar model
- describe the two-equal-groups problem in terms of halving
- use a known halving fact to solve the problem.

Work through several problems, including those where the cardinality *can* be seen and those where it *can't*. In *Example 2* opposite, it is obviously not possible for children to take the £20 note and split it into two; the note is simply representative of the quantity of money; you can use this as a discussion point with children, extending to consider how the amount in each money-box could be made up.

Example 1 – cardinality visible:

'There are two packets of dog bones. Each packet has the same number of bones. There are twelve bones altogether. How many bones are there in each packet?'



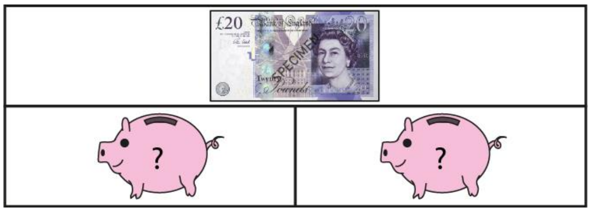
$2 \times \square = 12$ $\qquad$ $\square \times 2 = 12$

12	
?	?

- *'One of the factors is two, so the other factor is half of the product.'*
- *'Half of the bones are in each packet.'*
- *'I know that half of twelve is equal to six.'*
- *'So there are six dog bones in each packet.'*

Example 2 – cardinality *not* visible:

'Emily has £20 in total, kept in two money-boxes. Each money-box contains the same amount of money. How much is in each money-box?'



$2 \times \square = 20$ $\qquad$ $\square \times 2 = 20$

20	
?	?

- *'One of the factors is two, so the other factor is half of the product.'*
- *'Half of the money is in each money-box.'*
- *'Half of two tens is equal to one ten.'*
- *'So there is ten pounds in each money-box.'*

<table>
<tr>
<td>3:7</td>
<td>Now demonstrate that the generalisation can be applied to any problem where two is a factor and the other factor is unknown, even if '2' represents the group size, rather than the number of groups.</td>
<td>'There are twenty-six socks. How many pairs is this?'

$2 \times \square = 26$ $\square \times 2 = 26$

• 'One of the factors is two, so the other factor is half of the product.'

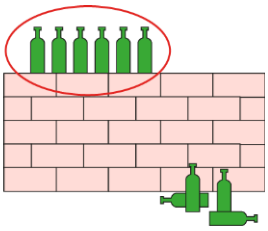
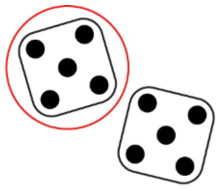
half of 26 = half of 20 + half of 6
= 10 + 3
= 13

• 'So there are thirteen pairs of socks.'</td>
</tr>
<tr>
<td>3:8</td>
<td>Provide children with varied practice, including:
• part–part–whole diagrams with missing numbers (like those in step 2:9)
• missing-number sequences, as shown opposite
• true/false style questions
• contextual problems, as shown on the next page and below:
  ◦ 'If I cut an eight metre piece of ribbon into two equal pieces, how long will each piece be?
  ◦ 'Last Friday I had half as many grapes in my lunchbox as I have today. Today I have twenty-two grapes. How many did I have last Friday?'
  ◦ 'I have ten pencils, which is half as many as Harry. How many pencils does Harry have?'
  ◦ 'Half the children on my table order one hot dog each. If there are three hot dogs, how many children are there?'
  ◦ 'I gave half of my toy cars to my friend; I have seven toy cars left. How many did I have to start with?'
  ◦ 'Felicity has twenty-eight marbles; she has twice as many marbles as Mahmud. How many marbles does Mahmud have?'</td>
<td>Missing-number sequences:
'Fill in the missing numbers.'
<table>
<tr><td>$2 \times 7 = 14$
Double 7 is equal to ___.
Half of ___ is equal to 7.</td><td>$6 \times 2 = 12$
Double 6 is equal to ___.
Half of ___ is equal to 6.</td></tr>
</table>

$2 \times 3 = 6$

$2 \times \square = 8$

$2 \times \square = 10$

True/false style problems:
'Tick the examples that represent half of 10.'
<table>
<tr><td> ☐</td><td>5 ☐</td></tr>
<tr><td>20 ☐</td><td> ☐</td></tr>
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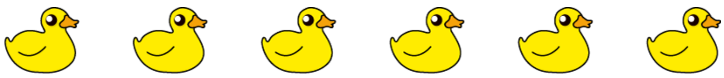
</table>

- *'If there are eight wheels in the bike-shed, how many bikes are there?'*

In each case, encourage children to draw a bar model to represent the information they are given.

Contextual problem:

'Half of the ducks are mine. How many ducks do I have?'



Dòng nǎo jīn:

★ + ★ = ♦

♦ + ♦ = 28

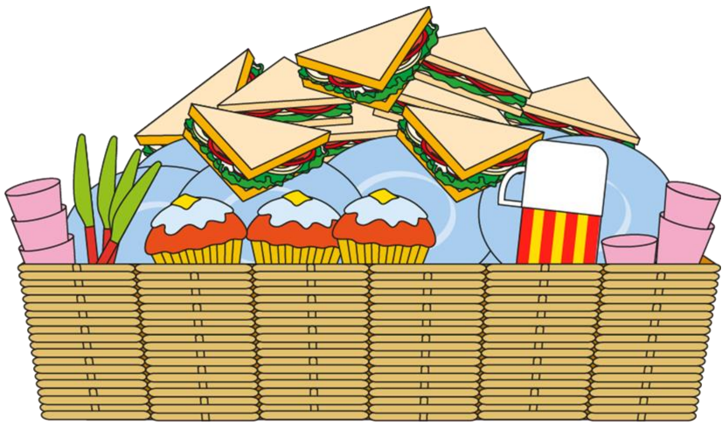
♦ = ☐ ★ = ☐

3:9

To round off *Teaching points 2–3*, give children opportunities to apply their knowledge of the two times table, doubling and halving in different contexts, such as the examples opposite.

Example 1:

'A shop sells hampers for people to take on picnics.'



'Fill in the gaps in the table.'

	Number in 1 picnic hamper	**Number in 2 picnic hampers**
cups	6	
knives		8
plates		12
cakes	3	
sandwiches	9	
flasks		2

Dòng nǎo jīn:

'How many of each item would there be in ***four*** *picnic hampers? '*

	Number in 4 picnic hampers
cups	
knives	
plates	
cakes	
sandwiches	
flasks	

Example 2:

'Count how many letters there are in each name.'

Li	Abby	Tom	Sadayo
Josh	Harrison	Hugo	Oliver
Georgina	Archie	Sara	Bazyli
Antonina	Adam	Mo	Arjuna

- *'Who has a name with twice as many letters as Li?'*
- *'Who has a name with double the number of letters in Abby?'*
- *'Who has a name with half the number of letters in Antonina?'*
- *'Who has a name with half the number of letters in Arjuna?'*

Teaching point 4:

Products in the ten times table are double the products in the five times table; products in the five times table are half of the products in the ten times table.

Steps in learning

4:1

Guidance

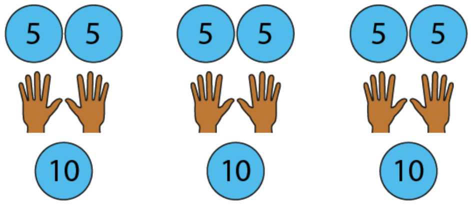
In segment *2.4 Times tables: groups of 10 and of 5, and factors of 0 and 1*, children looked at the relationship between the five and ten times tables. Here, the relationship is linked to the language and strategies of doubling and halving.

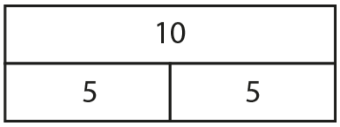
Briefly review some of what children learnt in segment *2.4*, *Teaching point 3*, including the following:

- Have half the class skip count in fives and half the class skip count in tens (supported by a hundred square or the Gattegno chart).
- The generalisations:
  - ***'For every one group of ten, there are two groups of five.'***
  - ***'Products in the ten times table are also in the five times table.'***
  - ***'Even multiples of five are also multiples of ten.'***

Representations

Relationship between groups of five and groups of ten:





'For every one group of ten, there are two groups of five.'

Comparing the five and ten times tables:

$0 \times 5 = 0$	$0 \times 10 = 0$
$1 \times 5 = 5$	$1 \times 10 = 10$
$2 \times 5 = 10$	$2 \times 10 = 20$
$3 \times 5 = 15$	$3 \times 10 = 30$
$4 \times 5 = 20$	$4 \times 10 = 40$
$5 \times 5 = 25$	$5 \times 10 = 50$
$6 \times 5 = 30$	$6 \times 10 = 60$
$7 \times 5 = 35$	$7 \times 10 = 70$
$8 \times 5 = 40$	$8 \times 10 = 80$
$9 \times 5 = 45$	$9 \times 10 = 90$
$10 \times 5 = 50$	$10 \times 10 = 100$
$11 \times 5 = 55$	$11 \times 10 = 110$
$12 \times 5 = 60$	$12 \times 10 = 120$

- *'Products in the ten times table are also in the five times table.'*

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<table>
<tr><td>0</td><td></td><td>0</td><td></td><td>0</td></tr>
<tr><td>2</td><td></td><td>10</td><td></td><td>1</td></tr>
<tr><td>4</td><td></td><td>20</td><td></td><td>2</td></tr>
<tr><td>6</td><td>× 5 =</td><td>30</td><td>= 10 ×</td><td>3</td></tr>
<tr><td>8</td><td></td><td>40</td><td></td><td>4</td></tr>
<tr><td>10</td><td></td><td>60</td><td></td><td>5</td></tr>
<tr><td>12</td><td></td><td></td><td></td><td>6</td></tr>
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'Even multiples of five are also multiples of ten.'
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<td>4:2</td>
<td>
Examine a pair of facts, for example:

$2 \times 5 = 10$
$2 \times 10 = 20$

Ask children what they notice, prompting them to describe how:

Ten is double five.
and
Two times ten is double two times five.
Five is half of ten.
and
Two times five is half of two times ten.

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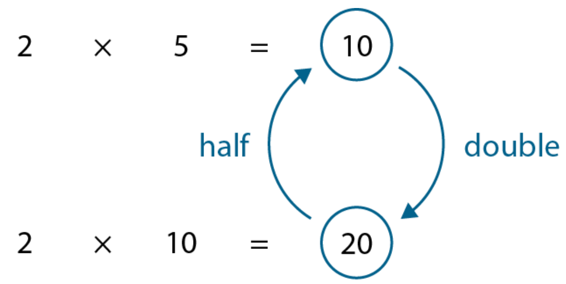

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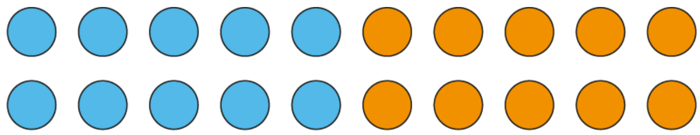
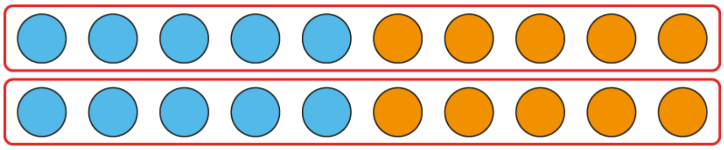
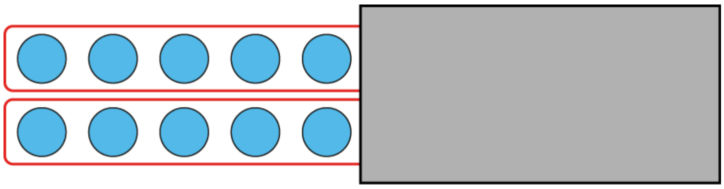
<td>4:3</td>
<td>Now look more closely at why the observation is true, using an array with two different counter colours to represent the equations examined in step 4:2. First focus on the group size of ten; draw around each group of ten and write the corresponding multiplication equation. Then halve the array (remove the counters of one colour, or cover them up as exemplified opposite); draw around the groups of five and write the corresponding multiplication equation. Use the following stem sentences to describe the relationships:

• 'Ten is double five, so ___ tens is double ___ fives.'
• 'Five is half of ten, so ___ fives is half of ___ tens.'

Then repeat, working systematically through the times table facts.</td>
<td>Two fives and two tens:

• 'How many tens are there?'

• 'There are two groups of ten.'
$2 \times 10 = 20$ 'Two tens are twenty.'
$10 \times 2 = 20$ 'Ten, two times is twenty.'

• 'How many fives are there?'

• 'There are two groups of five.'
$2 \times 5 = 10$ 'Two fives are ten.'
$5 \times 2 = 10$ 'Five, two times is ten.'
• 'Ten is double five, so two tens is double two fives.'
• 'Five is half of ten, so two fives is half of two tens.'</td>
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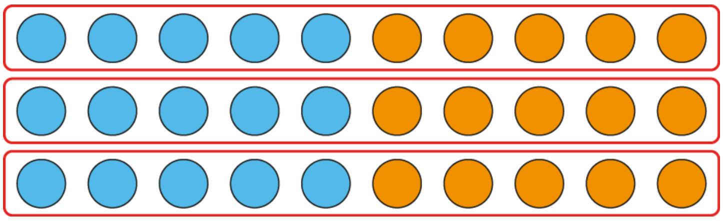
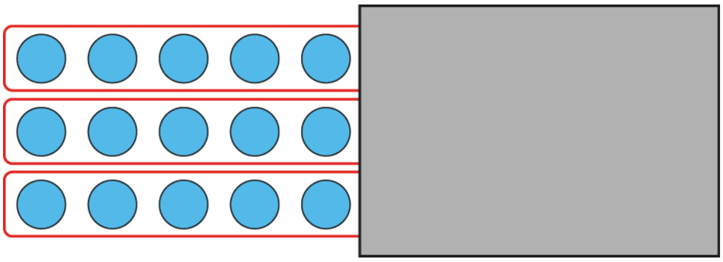
<td>
Three fives and three tens:

- *'How many tens are there?'*



- *'There are three groups of ten.'*
$3 \times 10 = 30$ *'Three tens are thirty.'*
$10 \times 3 = 30$ *'Ten, three times is thirty.'*

- *'How many fives are there?'*



- *'There are three groups of five.'*
$3 \times 5 = 15$ *'Three fives are fifteen.'*
$5 \times 3 = 15$ *'Five, three times is fifteen.'*
- *'Ten is double five, so three tens is double three fives.'*
- *'Five is half of ten, so three fives is half of three tens.'*
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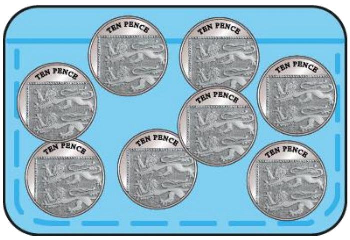
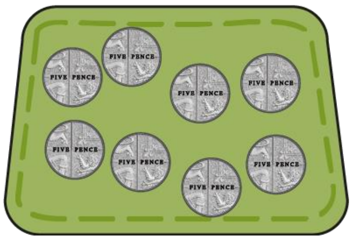
<td>4:4</td>
<td>Work through some contextual problems, such as the one shown opposite, using unitising language and known doubling/halving facts to find the unknown product.</td>
<td>
'Felicity has eight ten-pence coins; she has 80 p altogether. Martin has eight five-pence coins; how much money does Martin have altogether?'

Felicity



$8 \times 10 = 80$

Martin



$8 \times 5 = \square$

- *'Five is half of ten, so eight fives is half of eight tens.'*
- *'Half of eight is four, so half of eight tens is four tens.'*

$8 \times 5 = \boxed{40}$

- *'Martin has 40 p.'*
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<td>4:5</td>
<td>
Now shift the focus to cases where the product of a factor and five is equal to the product of a factor and ten (e.g., 2 × 5 = 1 × 10).
Use a familiar representation, such as hands and counters, as shown opposite. For a given number of hands (for example, four):

ask how many groups of five there are
represent the groups of five with five-value counters
represent the jumps of five on a number line
write a multiplication equation.

Then:

ask how many groups of ten there are
represent the groups of ten with ten-value counters
represent the jumps of ten on the same number line as the fives
write a multiplication equation.

Finally, compare the equations, asking children what they notice about the factors and the products.
Repeat for some other even quantities of hands, until the pattern is clear.
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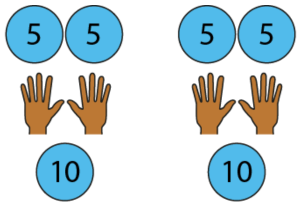
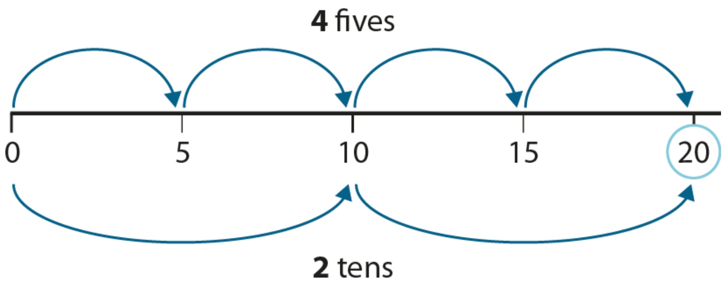


$4 \times 5 = 20$
$2 \times 10 = 20$


‘<u>Two</u> times ten is equal to twenty, so <u>double two</u> times five is equal to twenty.’
‘<u>Four</u> times five is equal to twenty, so <u>half-of-four</u> times ten is equal to twenty.’

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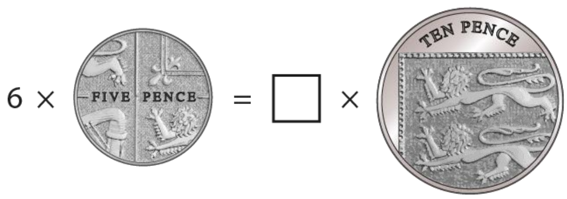
<td>4:6</td>
<td>
Building on the learning from step 4:5, work through some contextual problems, such as the one shown opposite, using known doubling/halving facts.
</td>
<td>
‘Olaf has six 5 p coins. Bo has some 10 p coins. Olaf and Bo have the same amount of money. How many 10 p coins does Bo have?’


$6 \times 5 = \square \times 10$

‘Six times five is equal to half of six times ten.’

$6 \times 5 = \boxed{3} \times 10$

‘Bo has three 10 p coins.’

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4:7

Provide children with practice based on steps *4:5* and *4:6* including:

- missing-number problems
- oral questions, such as:
  - *'How many fives equal four tens?'*
  - *'How many tens equal eight fives?'*
  - *'Ten tens is equal to one hundred, so twenty fives is equal to…?'*
- contextual problems, for example:
  - *'Amy has four 10 p coins. Jon has some 5 p coins. Jon has the same amount of money as Amy. How many coins does Jon have?'*
  - *'Five people can fit in one taxi. Ten people can fit in one minibus. There are eight full minibuses. If all of the people got out of the minibuses and into taxis, how many taxis would be needed?'*
  - *'Three children hold up their feet. They are showing three groups of ten toes. How many groups of five toes are they showing?'*

Missing-number problems:

'Fill in the missing numbers.'

$2 \times 10 = 20$

$2 \times 5 = \square$

$4 \times 10 = 40$

$4 \times 5 = \square$

$2 \times 5 = 1 \times 10$

$4 \times 5 = \square\ 2 \times 10$

$4 \times 5 = 2 \times 10$

$6 \times 5 = \square\ 2 \times 10$

$\square \times 5 = 2 \times 10$

$10 \times 5 = \square \times 10$

$60 = 10 \times 6$

$\square = 5 \times 6$

$90 = 10 \times 8$

$\square = 5 \times 9$

$20 \times 5 = \square \times 10$

$\square = 5 \times 9$

$40 \times 5 = \square \times 10$

$60 \times 5 = \square \times 10$

$\square \times 5 = 40 \times 10$

$\square \times 5 = \square \times 10$

Dòng nǎo jīn:

	True (✓) or false (×)
$8 \times 5 + 10 = 5 \times 10$	
$4 \times 10 > 9 \times 5$	
$3 \times 5 - 10 < 1 \times 10$	

♦ × ★ = 50

♦ × ♦ = 100

♦ = $\square$ ★ = $\square$

