8 Proportional reasoning

Mastery Professional Development

Solutions to exemplified key ideas

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Click the heading to move to that page. Please note that these materials are principally for professional development purposes; solutions are provided to support this aim. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

8.1.1.2 Understand how the zero property relates to a proportional relationship

Appreciate the zero-product property

Example 1:

All zero.

Example 2:

Responses may vary but should demonstrate an understanding that multiplying by zero always results in a product of zero. If one factor is zero, the product will be zero, whatever the other factors.

Appreciate the zero-product property in directly proportional contexts

Example 3:

a) Responses may vary but should demonstrate an understanding that this is not an example of a proportional situation. When there is one post, there are no planks. The zero-product property does not apply as there is no situation where both values are zero. The vertical posts need connecting so there will be more than 16 horizontal planks, doubling the posts does not double the planks.

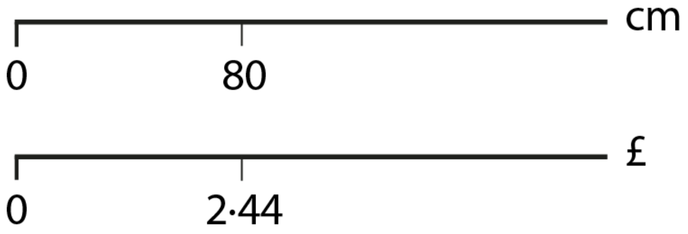
b) Responses may vary but should demonstrate an understanding that the situation does not involve direct proportion. The zero-product property does not apply since term number zero is not zero. Consequently, the tenth term cannot be multiplied by 10 to find the hundredth.

c) Responses may vary but should demonstrate an understanding that this is a multiplicative relationship and that for every 1 red counter, there are 3 stripey blue counters. If there were no red counters, there would be no stripey blue counters, so the zero-product property does apply.

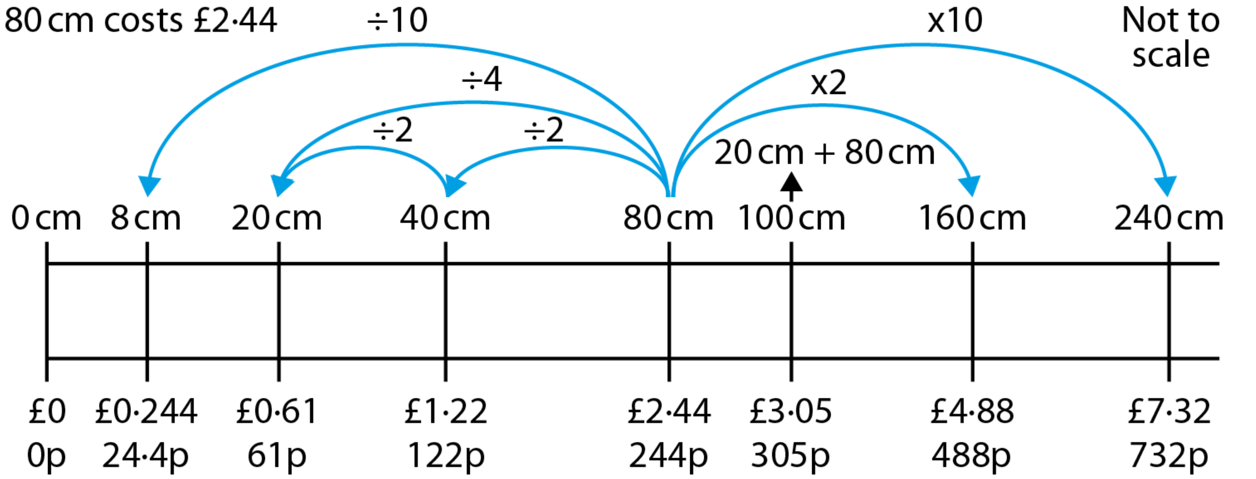
d) A key difference in the examples is the presence of a pair of zero values in a linear relationship distinguishing proportional situations from others. Responses may vary but should demonstrate an understanding that some sequences are proportional relationships between the position and the term, and others are not. It could be an interesting exercise to construct sequences where the 5th value can be doubled to find the 10th, but the sequence is not proportional.

Example 4:

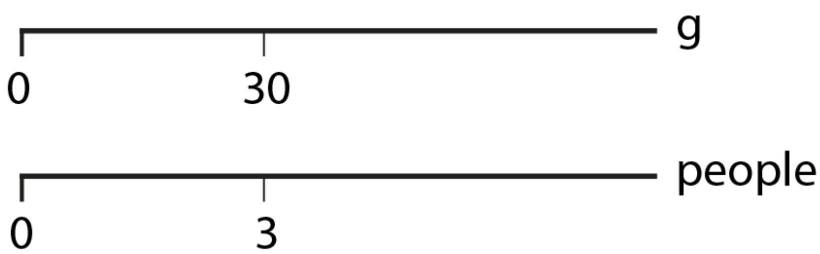
a)



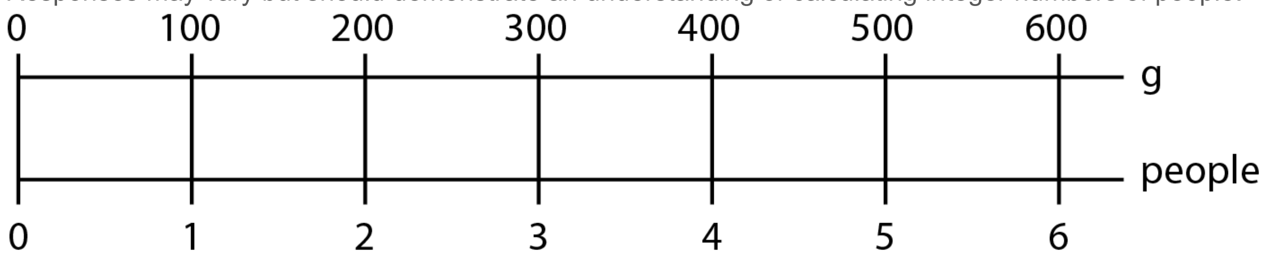
b) Responses may vary but should demonstrate an understanding that the cost of any quantity of ribbon can be calculated, but that some (for example, 8 cm) will cost a decimal number of pence.



c)



d) Responses may vary but should demonstrate an understanding of calculating integer numbers of people.



Recognise the features of directly proportional relationships in different representations

Example 5:

a) Responses may vary but might include statements such as, ‘Sixteen kilometres are equivalent to 10 miles’, ‘Four kilometres are equivalent to 2.5 miles’ and ‘Forty kilometres are equivalent to 25 miles’.

b) Representations may vary but will be a useful check for understanding. It is important that students are aware there are many correct representations, and that useful representations will make the zero-product property, constant of proportionality, and any scalar or functional multiplication visible.

8.1.1.5 Represent a proportional relationship algebraically

Appreciate the algebraic form of direct proportion, $y = kx$

Example 1:

a)

(i)
$y = total\ cost$
$x = litres\ of\ paint$
$y = 3x$

(ii)
$y = value\ in\ euros$
$x = value\ in\ pounds$
$y = 1.25x$

(iii)
$y = total\ distance\ travelled$
$x = number\ of\ hours\ spent\ travelling$
$y = 30x$

(iv)
$y = total\ cost$
$x = cost\ per\ item$
$y = px$

b)

(i) £12

(ii) $y =$ €3.125

(iii) $y = 4\ miles$

(iv) $y =$ £$7x$

Example 2:

	Red to blue	***Blue to red***	***Red to bag total***	***Blue to bag total***
$\_:\_$	$3:2$	$2:3$	$3:5$	$2:5$
fraction	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{2}{5}$
$y = kx$	$y = \frac{3}{2}x$ $y = red, x = blue$	$y = \frac{2}{3}x$ $y = blue, x = red$	$y = \frac{3}{5}x$ $y = red, x = total$	$y = \frac{2}{5}x$ $y = blue, x = total$

Example 3:

a) $y = 2x + 1$ No Responses may vary but should demonstrate an understanding that this graph does not pass through the origin.

b) $y = 2x$ Yes Responses may vary but should demonstrate an understanding that this graph passes through the origin and is linear.

c) $y = 2 + x$ No Responses may vary but should demonstrate an understanding that this graph does not pass through the origin.

d) $y = 2\sqrt{x}$ No Responses may vary but should demonstrate an understanding that this is not a linear graph.

e) $y = \frac{x}{2}$	Yes	Responses may vary but should demonstrate an understanding that the multiplier is $\frac{1}{2}$.
f) $y = \sqrt{2}x$	Yes	Responses may vary but should demonstrate an understanding that the multiplier is $\sqrt{2}$.
g) $y = x^2$	No	Responses may vary but should demonstrate an understanding that this is not a linear graph so y and x are not in direct proportion. There might be discussions around the relationship between y and x^2.
h) $y = x^2 + 1$	No	Responses may vary but should demonstrate an understanding that this graph does not pass through the origin.
i) $y = 2^x$	No	Responses may vary but should demonstrate an understanding that this is not a linear graph.
j) $y = 2x^3$	No	Responses may vary but should demonstrate an understanding that this is not a linear graph. Further discussions might be had, as in part g.

Recognise where direct proportion exists in non-linear relationships

Example 4:

Radius	Radius squared	Area
1	1	π
2	4	4 π
3	9	9 π
4	16	19 π
5	25	25 π

Responses may vary but should demonstrate an understanding that Anne is correct as the constant of proportionality is π so the graph this produces is $y = \pi x$ (where $x = r^2$ and $y = \pi r^2$)

Understand the algebraic form of inverse proportion $y = \frac{k}{x}$

Example 5:

a) Responses may vary but should demonstrate an understanding that, in each row, the value of the shaded fractions will decrease but the whole remains the same.

b) Responses may vary but should demonstrate an understanding that the denominators increase by 1 each row.

Example 6:

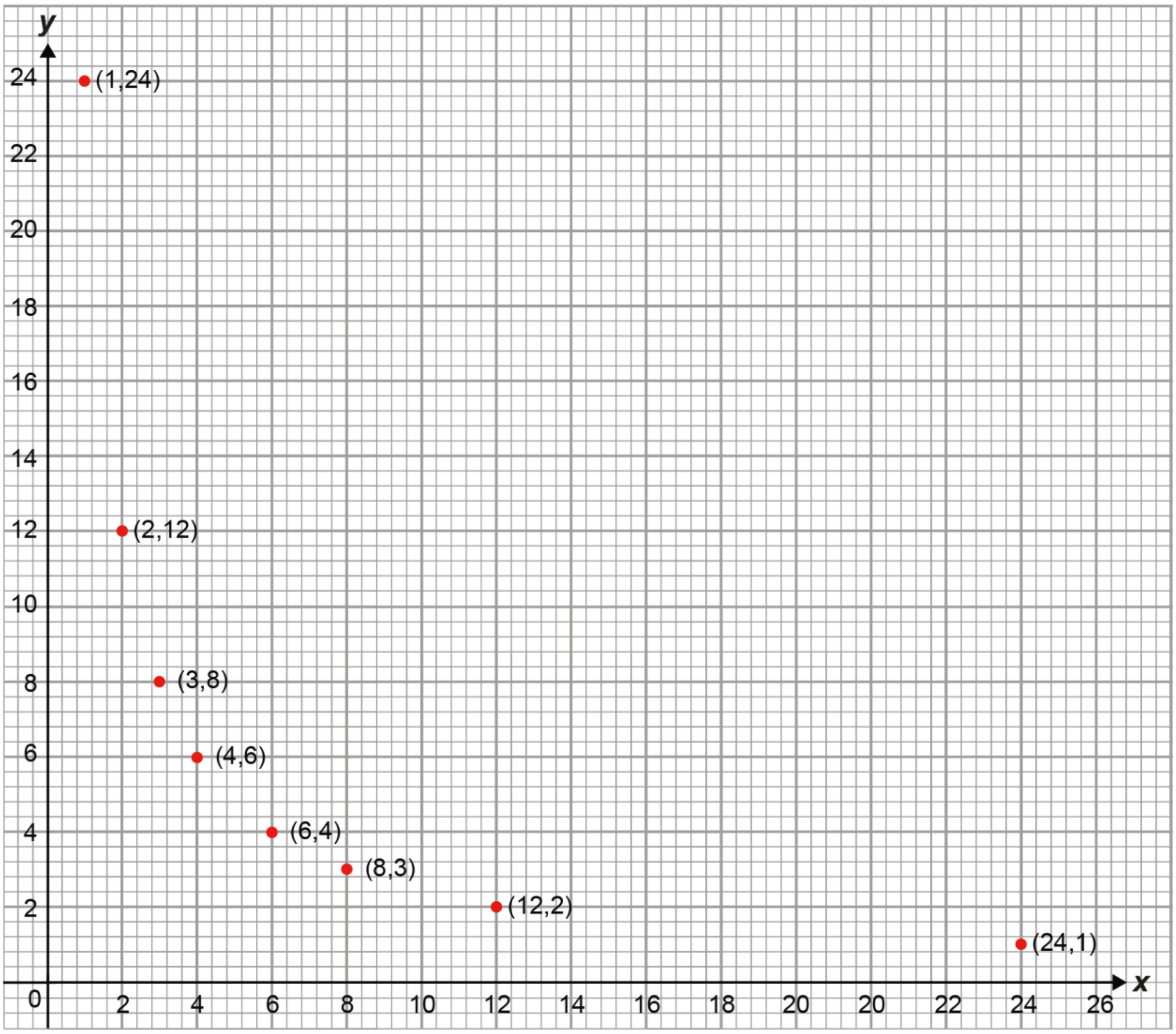
a) Responses may vary but should demonstrate an understanding that the dimensions could be any two numbers with a product of 24.

b) As for part a.

c) 1×24 and 2×12; 2×12 and 4×6; 3×8 and 6×4

d) $y = \frac{24}{x}$

e) Responses may vary but a sample graph is provided below. Students may notice the graph resembling a reciprocal graph.



Example 7:

Responses may vary but could include: $y \propto \frac{1}{x}$, $y = \frac{k}{x}$, $12 = \frac{k}{4}$ and graphs or tables demonstrating this relationship.

8.1.2.3 Understand that compound measures are a multiplicative comparison between two quantities of different types

Understand the notion of 'per' and 'for every'

Example 1:

a) 770 g
For every person, 110 g of flour is required.

b) $793.\dot{3}$ g
For every person, $113.\dot{3}$ g of flour is required.

c) 240 g

For every person, 80 g of flour is required.

Example 2:

a) Responses may vary but should demonstrate an understanding that for every 1 kg, there are $1.5\ m^3$ of the packaging, and/or that every 1m^3 of cardboard packaging weighs $\frac{2}{3}$ kg

b) Responses may vary but the values found should demonstrate an understanding of the proportionality: for every 1 kg, there are $\frac{10}{3}\ m^3$ of airbag packaging and that every 1m^3 of airbag packaging, weighs 0.3 kg.

Understand the general form of compound measures $\boldsymbol{k = \frac{y}{x}}$

Example 3:

a) $\frac{9.58}{100} = 0.0958$ seconds per metre.

b) $\frac{100}{9.58} = 10.4$ m/s (to 1dp).

c) $\frac{46.80}{100} = 0.4680$ seconds per metre.

d) $\frac{100}{46.86} = 2.1$ m/s (to 1dp).

e) Responses may vary but could involve ideas around the faster speed having the higher value; that distances per unit time fit with the lived experience, and that this leads more naturally to measures for acceleration. The conversation could involve pace as an inverse of speed, as well as the need for units to be practical and changeable by convention.

Example 4:

Representation	***Where is 165?***	***Where is 3?***	***What does 55 mean?***
Graph	The point $(3, 165)$ shows that after 3 hours, the distance is 165 miles	The point $(3, 165)$ shows that after 3 hours, the distance is 165 miles	For every 1 hour of travel, there is a 55-mile increase in distance. Shown as the gradient.
Double number line	Fourth dash on the “miles” line	Fourth dash on the “hours” line	The number of miles travelled in 1 hour. This could be shown as the functional multiplier from hours to miles.
Ratio Table	The difference between 55 and 220 miles	The difference between 1 and 4 hours	The number of miles travelled in 1 hour
The average speed for all is 55 miles per 1 hour.			

8.2 Understanding graphical representations of proportionality

8.2.1.2 Understand that a repeated percentage change can be calculated through repeated use of a single multiplier

Understand that, in repeated percentage changes, the 'whole' changes at each stage

Example 1:

a) Responses may vary but could include 'rectangle B is 50% of rectangle A', or 'thinking about rectangle A, B is 50%' to emphasise the subject of comparison.

b) C is 50% of rectangle B. D is 50% of rectangle C.

c) Rectangle C is 25% of rectangle A. Rectangle D is 25% of rectangle B. Rectangle D is 12.5% of rectangle A.

d) Responses may vary but should demonstrate an understanding that Mo is correct because rectangle D is 50% of the area of rectangle C, which is 50% of the area of rectangle B, which is 50% of the area of rectangle A, which is 50% of the total area of the whole rectangle.

Example 2:

a) Responses may vary but should demonstrate an understanding that a 50% reduction each day means there is a new starting price each day. Ken was thinking about reducing by subtracting 50% of the original amount each day.

b) After two days the reduction is 75%; after three days, 87.5% reduction.

Example 3:

a) Four

b) Responses may vary but should demonstrate an understanding that the starting number will make no difference to the rate of doubling.

c) $M \rightarrow 1.2M \rightarrow 1.44M \rightarrow 1.728M \rightarrow 2.0736M \rightarrow \dots$

Responses may vary but should demonstrate an understanding that M has doubled when the coefficient of M is greater than 2.

Example 4:

a) 7 cm

b) 7 weeks.

c) 20%

d) Shorter, at 5 cm.

e) Responses may vary but might note that the flowers have the same rate of growth, 20%.

f) Yes, for 9 weeks.

g) The index, 9, indicates that the percentage increase happens 9 times.

Example 5:

Completed cells are given in **bold**.

In words	Expression	Original quantity	Percentage change	Number of repetitions
Increase 27 by 15%, 3 times.	27×1.15^3	27	15% increase	3
Increase 30 by **81**%, **8** times.	30×1.81^8	**30**	**81% increase**	**8**
Increase 12 by 16%, 12 times	$\mathbf{12 \times 1.16^{12}}$	12	16% increase	12
Decrease 17 by 25%, 4 times	17×0.75^4	**17**	25% decrease	**4**
Decrease 106 by 10%, 6 times.	$\mathbf{106 \times 0.9^6}$	**106**	**10% decrease**	**6**
Decrease 77 by 23%, 7 times	77×0.77^{47}	**77**	**23% decrease**	**7**

8.2.2.3 Know how the key features of a proportional relationship are represented graphically

Understand that the gradient of a straight-line graph that goes through the origin represents the constant of proportionality

Example 1:

Responses may vary but should demonstrate an understanding that B and D represent the proportional relationship $y = 2x$, because in both graphs the point shown is $(x, 2x)$ and the graph goes through the origin. Graph A shows an x value twice the y value and graph C does not show a directly proportional relationship as the line is not going through the origin.

Use graphical representations to distinguish between relationships that are directly proportional and non-proportional linear relationships

Example 2:

B: $y = \frac{1}{2}x$ F: $y = 2x$ H: $y = -2x$

Be familiar with the graphical representations of standard proportional relationships

Example 3:

A: $y \propto x^2$ B: $y \propto \sqrt{x}$ C: $y \propto x^3$ D: $y \propto x$

Example 4:

E: $y \propto \frac{1}{x^3}$ F: $y \propto \frac{1}{\sqrt{x}}$ G: $y \propto \frac{1}{x^2}$

Appreciate the underlying multiplicative structure of proportional relationships $\boldsymbol{A = kB}$ where $\boldsymbol{B = x^n, x > 1}$

Example 5:

Responses may vary but should demonstrate an understanding that Shaun only considers the relationship between y and x , not x^2; while Sarah demonstrates the proportional relationship between y and x^2.

8.2.2.4 Interpret the gradient as a rate of change

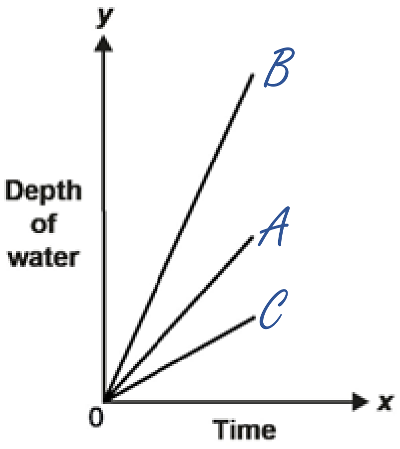
Understand the relationship between the constant of proportionality and the gradient for directly proportional linear relationships

Example 1:

A: $y = 2x$ L   B: $y = 3x$ K   C: $y = \frac{1}{2}x$ M   D: $y = \frac{x}{4}$ N

Appreciate the rate of change as a description of how one quantity changes in relation to another

Example 2:



Understand that the gradient is equal to the unit rate

Example 3:

The difference in price is 35p per litre.

Determine what the rate of change represents for a variety of contexts

Example 4:

A: The distance per unit time.
Speed in km per hour.

B: The speed per unit time.
Acceleration in metres per seconds^2

C: The mass per unit volume.
Density in grams per cm^3

Identify negative rates of change *Example 5:* Responses may vary but should demonstrate an understanding that: A is the only graph shown in the 4th quadrant B is the only graph representing a non-proportional linear relationship C is the only graph with a positive rate of change D is the only graph shown in the 2nd quadrant.
Interpret negative rates of change *Example 6:* The rate of temperature change is $\frac{-5}{6}°C$ per hour. The temperature is falling at a rate of $\frac{5}{6}°C$ per hour.

8.2.3.4 Interpret the gradient of the tangent to a curve as the rate of change of the relationship represented by the curve at that instant

Appreciate that the gradient of a curve constantly changes *Example 1:* Responses may vary but should demonstrate an understanding that the gradient is constantly changing because this is a curve. Edward interpreted the coefficient as the constant of proportionality and Esther found the gradient at a particular point.
Understand that the gradient of the tangent to a curve provides an estimate for the rate of change *Example 2:* Responses may vary but should demonstrate an understanding that Solomon and Stacey used different values to calculate their rate of change, resulting in different answers because the calculated gradient is an estimation.
Know that a tangent can be drawn at any point on the graph and gives the rate of change of the curve at that point *Example 3:* a) Responses may vary but should demonstrate an understanding that the rate of change can be calculated using the quotient of the changes in x and y (shown below) and described as either a rate of change of -0.95 cm/second or that the depth decreases at a rate of 0.95 cm/second. $m = \frac{5-9}{6-1.8} = \frac{-4}{4.2} = -0.95\ (to\ 2dp)$ b) After approximately 3.6 seconds. c) Responses may vary but should demonstrate an understanding that the rate of change can be calculated at any point where a tangent can be drawn.