7 Using and applying numerical structure

Mastery Professional Development

7.1 Using structure to calculate and estimate

Guidance document | Key Stage 4

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The first of the Key Stage 4 themes (the seventh of the themes in the suite of Secondary Mastery Materials) is *Using and applying mathematical structure*, which covers the following interconnected core concepts:

7.1 Using structure to calculate and estimate

7.2 Using structure to transform and evaluate expressions

This guidance document breaks down core concept *7.1 Using structure to calculate and estimate* into four statements of **knowledge, skills and understanding**:

7.1 Using structure to calculate and estimate

- 7.1.1 Work with estimated values
- 7.1.2 Calculate with numbers expressed in standard form
- 7.1.3 Work interchangeably with recurring decimals and their corresponding fractions
- 7.1.4 Use structure to estimate solutions

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

- 7.1.1 Work with estimated values
  - 7.1.1.1 Understand the difference between rounding and truncating
  - 7.1.1.2 Identify limits of accuracy when rounding or truncating
  - 7.1.1.3 Use and interpret upper and lower bounds appropriately in calculations
- 7.1.2 Calculate with numbers expressed in standard form
  - 7.1.2.1 Understand the mathematical structures that underpin multiplication and division of numbers represented in standard form
  - 7.1.2.2 Understand the mathematical structures that underpin addition and subtraction of numbers represented in standard form
- 7.1.3 Work interchangeably with recurring decimals and their corresponding fractions
  - 7.1.3.1 Understand the infinite nature of recurring decimals
  - 7.1.3.2 Convert between a recurring decimal and a fraction

7.1.4 Use structure to estimate solutions

- 7.1.4.1 Understand that the relationship between a number and its powers and roots is not linear
- 7.1.4.2 Understand that known facts about squares and cubes can be to estimate other values
- 7.1.4.3 Find approximate solutions to equations numerically using iteration

Overview

This core concept is concerned with understanding the structure of numbers and the number system. There is a particular focus on place value and manipulating the way that numbers are represented so that their structure can be used to efficiently evaluate calculations and make estimations. The intention is for students to achieve fluency in identifying and manipulating mathematical structures so that estimations of calculations can be carried out elegantly and with minimal effort.

Throughout, students are building on knowledge that was first introduced in Key Stages 2 and 3. Students need to have established a secure understanding of the structures involved so that they can work more deeply with them and use them to make new connections.

For example, at Key Stage 3, students learnt to express very large numbers using standard form, and to write numbers as the product of factors. They also learnt about the equivalence of fractions and decimals. Now, at Key Stage 4, this core concept exploits their proficiency, so that they can flexibly and fluently manipulate a number into a form that is useful when solving a problem that involves calculation with very large or small numbers. This means that students can consider efficient solutions and make choices in their mathematics, based on deeper conceptual understanding rather than a narrow understanding of procedure.

Two aspects of estimation are explored in this core concept. The first involves limits of accuracy, building towards the use of upper and lower bounds within calculations. The second looks at estimation of squares and roots, leading to iterative processes for estimating solutions. Again, we suggest a structural approach will best support students to deepen their understanding of these ideas. For example, exploring the relationship between a number and its powers, perhaps through different representations, will help students to appreciate its non-linearity. This helps them to appreciate that the square of a number that is midway between two other numbers will not be midway between their two squares, leading to them being able to estimate more accurately.

Throughout the professional development materials, numeric and algebraic concepts have been presented together so that students can recognise that the underlying structures are the same. This is particularly salient when working with recurring decimals and their equivalent fractions, as they will use their knowledge of manipulating and solving algebraic equations to convert from decimals to fractions. Prior to this understanding, students need a solid sense of the infinite nature of recurring decimals, which includes an appreciation that writing the quantity as a decimal requires truncation, and so its fraction form is technically a more accurate representation of the number.

A key challenge throughout the mathematics of theme 7 is perceiving the value represented as both an object and a process. For example, $\frac{1}{9}$ can be thought of as both the process of dividing 1 by 9 and as the result of that process. Similarly, 3×10^5 is a way of representing the product of 3 and 10^5 and a way of writing the 'answer'. This object/process duality is sometimes referred to as a procept and is described further in the overview for PD materials '*7.2 Using structure to transform and evaluate expressions'*.

Prior learning

Students' understanding of the structure of number has been building from their very earliest work in Key Stage 1. By the end of Key Stage 2, they should have a solid understanding of decimal and integer place value, including be able to round values to the nearest specified place value. At Key Stage 3, this was developed further to include several other mathematical structures for rounding. For example, students should now be familiar with rounding to decimal places and significant figures (although they are unlikely to have met truncation before). They should also have started to think about errors and over/under-estimates, in preparation for work on upper and lower bounds in calculations at Key Stage 4.

At Key Stage 3, students also developed their understanding of the different ways that numbers can be expressed and built their proficiency in changing numbers from one form to another to reveal different structures.

Students should know that a number can be expressed multiplicatively (for example, the number 231 can be expressed as the product 3 × 7 × 11, or as 33 × 7, 21 × 11, etc). This multiplicative understanding, combined with their understanding of the structure of place value, should have led to an awareness that numbers can be expressed in standard form, i.e. as the product of a power of 10 and a value between 1 and 10.

Students will also have an understanding that a number can be expressed additively (for example, the number 231 can be expressed as the sum 200 + 30 + 1, or as 100 + 131, or in a multitude of other forms) and that elements of the same magnitude can be added easily. This links to the idea of 'unitising', namely that numbers of the same 'unit' can be combined, which will be built on further in students' work on calculating with standard form and with their work on indices and surds in *7.2 Using structure to transform and evaluate expressions*.

Similarly, students' work on fractions will have its foundations in their primary education, both in terms of their conceptual understanding of what a fraction is and their procedural understanding of calculating with fractions. This should have developed in tandem with fluency in converting between terminating decimals and their fraction equivalents, so that they have an appreciation that both are representations of the same value and are comfortable using either when appropriate. This understanding, alongside a familiarity with both written methods for division and manipulating algebraic equations, are the key pre-requisites for converting between recurring decimals and their fraction equivalents. The Key Stage 3 Professional Development materials focused heavily on understanding the structure of fractions calculations, so that work at Key Stage 4 can build upon generalising this understanding. In the case of this core concept, division of numbers written in standard form will also be much easier for students to access if they have fluency with identifying common factors and using these to simplify fractions.

Students' work on estimating solutions in key idea 7.1.4 builds on several key understandings from Key Stage 3. Firstly, they need to have an appreciation of what a solution is, which will have developed through their work on manipulating equations and on linear graphs. They will also have worked extensively on square and, to a lesser extent, cube numbers, so should be familiar with the sequence of these numbers as well as the relationship between roots and powers. Making connections to graphical representations will be useful here.

All four core concept documents within *1 The structure of the number system* from the Key Stage 3 PD materials explore the prior knowledge required for this core concept in more depth.

Checking prior learning

The following activities from the NCETM secondary assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Secondary assessment materials page 8	Fill in the box to make the calculations correct. $3 \times 10^4 \times [\quad] = 6 \times 10^5$ Explain your method.

Secondary assessment materials page 8	If 6 miles ≈ 10^4 metres, which of these is approximately 600 miles? a) 10^{10} metres b) 10^6 metres c) 10^8 metres Explain your reasoning.
Key Stage 3 PD materials document '*1.3 Ordering and comparing*', Key idea 1.3.3.1, Example 3	a) Complete this family of equations by filling in the gaps. (i) $0.3 \times [\quad] = 1200$ (ii) $3 \times 400 = 1200$ (iii) $30 \times 40 = 1200$ (iv) $300 \times [\quad] = 1200$ (v) $[\quad] \times 0.4 = 1200$ b) How would you calculate $0.0003 \times 4\,000\,000$?
Key Stage 3 PD materials document '*1.3 Ordering and comparing',* Key idea 1.3.3.1, Example 6	Work out: a) 6.7×100 b) $\frac{67}{10} \times 100$ c) 6.7×10^2 d) 34×0.01 e) $34 \times \frac{1}{100}$ f) $34 \times \frac{1}{10^2}$

Key vocabulary

Key terms used in the Key Stage 3 Professional Development materials

- associative
- cube root
- distributive
- exponent
- square root
- standard index form
- terminating decimal

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
iteration	The process of repeating a calculation or algorithm until a result of a certain degree of accuracy is obtained. Iteration can be used to estimate approximate solutions to calculation.

lower/upper bound	The lower and upper bounds are, respectively, the smallest and largest values that a number could have taken prior to being rounded. They represent the range, or the limits of accuracy, of possible values of a rounded or truncated value.
recurring decimal	A decimal fraction with an infinitely repeating digit or group of digits. Example: The fraction $\frac{1}{3}$ is the decimal 0.33333 ..., referred to as nought point three recurring and may be written as $0.\dot{3}$ (with a dot over the three). Where a block of numbers is repeated indefinitely, a dot is written over the first and last digit in the block, for example $\frac{1}{7} = 0.\dot{1}4285\dot{7}$.
truncation	An alternative to rounding as a way of approximating a number. When a number is truncated, it is shortened at a specified point, with placeholder zeroes used to maintain place value. Unlike rounding, the digits do not change, regardless of whether the next digit in the number is greater than or equal to five. Example: 3 529 truncated to one significant figure is 3 000, whereas it would be 4 000 when rounded to one significant figure.

Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches, and offer prompts to support professional development and collaborative planning.

7.1.1 Work with estimated values

Students need to understand that there are situations where a compromise can be made about the degree of accuracy in order to solve a problem efficiently. Estimation in this context is not about algorithmically rounding values to a given number of significant figures before calculating. Rather, it is about developing a sense of the structures of both numbers and operations, to identify how decisions around accuracy will impact on the calculation.

Working with limits of accuracy requires students to understand the range of numbers that round to a particular value, and how this range changes as the degree of accuracy changes. As students move towards choosing and using bounds in calculation, context becomes ever more important. Students should, for example, appreciate the difference between continuous and discrete values.

7.1.1.1 Understand the difference between rounding and truncating

7.1.1.2 Identify limits of accuracy when rounding or truncating

7.1.1.3 Use and interpret upper and lower bounds appropriately in calculations

7.1.2 Calculate with numbers expressed in standard form

At Key Stage 3, students met standard form to represent large and small quantities. As they move to Key Stage 4, they use this representation to calculate efficiently with numbers of magnitude.

Students should appreciate that standard form is a factorisation of a number $(2.4 \times 10^5$ is stating that 240 000 can be written as the product of 2.4 and 100 000). This, along with an understanding of associativity, underpins strategies used when multiplying and dividing numbers in standard form.

When calculating additively with numbers written in standard form, it is helpful to make links to the unitising that has underpinned so much work in algebra at Key Stage 3 and number at Key Stages 1 and 2. Students need an understanding of the distributive law combined with an awareness that addition and subtraction of numbers in the same 'unit' can be easily carried out. For example, 3 hundreds added to 4 hundreds is 7 hundreds, and 3 ten thousands added to 4 ten thousands is 7 ten thousands. They must understand and fluently manipulate the form of the numbers, so that their factors are of the same magnitude, to carry out an additive calculation.

7.1.2.1 Understand the mathematical structures that underpin multiplication and division of numbers represented in standard form

7.1.2.2 Understand the mathematical structures that underpin addition and subtraction of numbers represented in standard form

7.1.3 Work interchangeably with recurring decimals and their corresponding fractions

A key challenge when working with recurring decimals is the understanding of their infinite nature when represented as a decimal number. Here, students have an opportunity to work with, and consider the limits of, the place-value structure as a way to represent numbers easily.

The manipulations used when converting a fraction to a recurring decimal rely on this understanding of infinity and so can feel counterintuitive to students. Using the formal written method for division to offer an insight into the iterative nature of calculating the next decimal place may help. Students will then connect this understanding to their existing algebraic fluency to convert recurring decimals to fractions.

7.1.3.1 Understand the infinite nature of recurring decimals

7.1.3.2 Convert between a recurring decimal and a fraction

7.1.4 Use structure to estimate solutions

Here, students are working with ‘solutions’ in a different way to their work on algebra at Key Stage 3, so they need to be comfortable with the idea that solutions do not have to be integers, fractions or terminating decimals, and that it is possible to estimate a solution through iteration. Working with the familiar concepts of squares and cubes offers a way into more estimating and iterating solutions to more complex equations. Students should consider the non-linear relationship between numbers and their powers and roots, which might be explored through representations such as dynamic geometry (to show areas and side lengths), number lines or graphs.

7.1.4.1 Understand that the relationship between a number and its powers and roots is not linear

7.1.4.2 Understand that known facts about squares and cubes can be used to estimate other values

7.1.4.3 Find approximate solutions to equations numerically using iteration

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have, and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

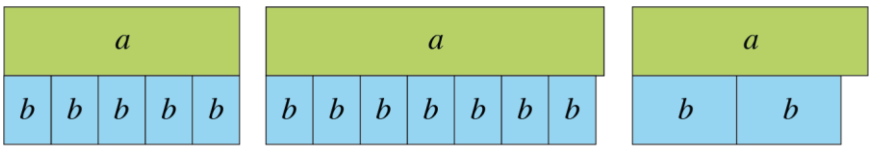
These are indicated by this symbol.

7.1.1.3 Use and interpret upper and lower bounds appropriately in calculations

Common difficulties and misconceptions

Arguably the most common issue when students work with upper and lower bounds is the misconception that choosing the maximum values to work with will always result in the maximum possible answer (and vice versa). While this works for positive numbers for addition and multiplication, this is not necessarily the case for subtraction and division, and particular consideration needs to be given to cases involving negative numbers. Students need to have a deep understanding of the structures of these calculations, and to explore the effect of changes to different values. Bar models and number lines can be useful representations for visualising this impact, particularly if these are demonstrated dynamically.

For example, the image below demonstrates the effect of increasing the value of the dividend compared with increasing the value of the divisor:



Students also need to be aware of the difference between continuous and discrete measures, and when they can use certain values. For example, if the limits of accuracy of a value are $250 \leq x < 350$, they need to be aware that the upper bound would be 349 in a discrete context (such as the number of items in a box) but that it's appropriate to use 350 when calculating in a continuous context (such as the weight of a box).

Students need to	Guidance, discussion points and prompts
Work confidently with different degrees of accuracy *Example 1:* *Match which of following will have the same answer when you round 81972.3564 to:* *2 significant figures* *2 decimal places* *The nearest 1 000* *The nearest 100* *The nearest tenth* *The nearest hundredth* *5 significant figures* *5 decimal places* *The nearest million* *The nearest integer* *The nearest 100 000* *3 significant figures*	*Example 1* explores students' understanding of the different **language** used to describe degrees of accuracy, so that they make connections between them. It is important that students understand that a phrase such as 'decimal places' can always be referred to using the same place-value description, but that 'significant figures' are determined by the magnitude of the number. In particular, draw attention to: • why, in this particular case, rounding to 2 and 3 significant figures produces the same result; • how students deal with 5 decimal places, and whether they consider their result to be the same as if the statement was 4 decimal places; • what other ways they could express the statements that do not have a matching pair. Reflect on your schemes of learning. What language do you use to refer to rounding at various stages in the curriculum? Are explicit links made between the language that you introduce over Key Stages 3 and 4, and the language that might have been used in Key Stage 2?
Example 2: *An elephant's weight is estimated to be approximately 5 000 kg.* *Paul says that a possible weight for the elephant is 5 443 kg.* *Would he be right if the estimate was correct to:* *a) The nearest 10 kg?* *b) The nearest 100 kg?* *c) The nearest 1 000 kg?* *d) The nearest 10 000 kg?* *e) 1 significant figure?* *f) 2 significant figures?*	Students should already have considered limits of accuracy (as described in key idea 7.1.1.2) and this task offers an opportunity for **deepening** this understanding. Rather than offering a range of possible values for a rounded number, students have to consider when a particular value could be possible. Ask students to consider what is the same and different about parts *a* to *d* and *e* to *f*, and whether their answers would be the same if the values were 500 kg and 544.3 kg respectively.

Identify upper and lower bounds in both continuous and discrete contexts

Example 3:

A tank is manufactured to have a capacity of 500 litres, correct to the nearest 10 litres.

a) What is the maximum amount of liquid that might be able to be stored in the tank?

A barrel holds 500 apples, correct to the nearest 10 apples.

b) What is the maximum number of apples that might be able to be stored in the barrel?

The **variation** in *Example 3* is straightforward: the structure of the questions is identical, as are the numbers used, but the context is varied. This draws attention to what is different about the two contexts – namely, that one involves continuous measurement (liquid) and the other discrete (numbers of apples).

Consider the **language** that you use in this and other examples with students. For example, these examples rely on a familiarity with the word ‘capacity’. Is this a word they will have been exposed to in their mathematics experience? In what other contexts might they have come across this vocabulary? Rather than avoiding certain vocabulary, be aware of when it might be a barrier and draw attention to potential unfamiliar words and their meanings. This can support students to access worded and context-based questions.

Example 4:

Pip buys two planks of wood, each measuring 2 m to the nearest cm.

She tries to work out the maximum total length of wood using this calculation:

$$\begin{array}{r} 2.0049999999999 \\ +\ 2.0049999999999 \\ \hline 4.0099999999998 \\ {\scriptstyle 1\,1\,1\,1\,1\,1\,1\,1\,1\ \ } \end{array}$$

How would you persuade Pip to do her calculation more efficiently?

As with *Example 3*, *Example 4* grapples with continuous data and, in particular, how to handle the upper bound. It asks students to consider a particular **representation** of the number $2.004\dot{9}$ and whether it is an appropriate way to use it in calculations. It is an open question, so conversations with students might range across what level of accuracy is required in different contexts; what is practical to do when operating on number; and the thorny issue of how to deal with irrational numbers and infinity in mathematics.

Discuss with colleagues how you address this with students. You might, for example, draw on their knowledge of converting recurring decimals to fractions to prove that $0.\dot{9}$ is equivalent to 1. Or use more informal justifications about the difference between $0.\dot{9}$ and 1 being so small as to be insignificant. Do you use the same approach with every class? Why or why not?

Recognise the effect of increasing or decreasing a value in different calculations

Example 5:

Sahar thinks of three different positive numbers: a, b and c such that $a < b < c$

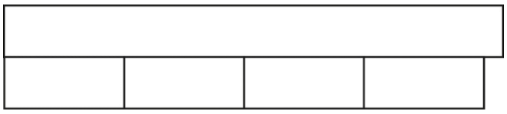
She thinks of a fourth positive number, d.

*Which of a, b and c should she choose to complete these calculations, so that it each has the **biggest** possible solution?*

$d - [\_]$	$[\_] \times d$
$d + [\_]$	$d \times [\_]$
$[\_] - d$	$d \div [\_]$
$[\_] + d$	$[\_] \div \boldsymbol{d}$

Working appropriately with upper and lower bounds in calculations requires students to have a secure understanding of the structure of the four operations. This task focuses on **deepening** this understanding, without including the context of limits of accuracy. The focus is purely on how the calculation is affected by choosing a bigger or smaller number.

You may find that **representations** such as Cuisenaire® rods represent the three different numbers a, b and c help students to think of this task in a structural way. Using a number line may also be helpful and allows for students to consider that any of the numbers may also be negative.

Example 6: *This bar model represents the calculation* $125 \div 30$.  *Given that* $120 \div 30 = 4$, *describe how the bar model would change if:* *a) 30 increased to 35* *b) 30 decreased to 25* *c) 30 increased to 30.5* *d) 30 decreased to 29.5.*	Bar models can be a useful **representation** for explaining which bound to use, particularly for division where students might instinctively (and incorrectly) choose the 'biggest' to get the 'biggest' answer. As with *Example 5*, there is no reference to upper and lower bounds here. Instead, the focus is on the structure of the calculation and how changing one number would affect the outcome. The **variation** in this question is designed so that the focus is on the divisor and considering how a change in its value affects the quotient. This is intended to lead into a discussion of how the selection of the upper or lower bounds for the divisor will impact upon the magnitude of the quotient. Note that, throughout, the 125 remains constant and so the size of the top bar remains the same. Discuss this example with colleagues. How would it change if the dividend (i.e. the 125) was varied, rather than the divisor?
Example 7: *Chris says he has come up with a rule for finding the upper bound for any calculation involving only positive numbers. He says, 'I just need to use the biggest possible value each time.'* *For which of the calculations a to f below will Chris's rule work, and for which will it not work?* *a)* $x + y$ *b)* $x - y$ *c)* $x \div 2y$ *d)* $x^2 \times y$ *e)* $\frac{y}{2} - x$ *f)* $xy - \frac{x}{y}$	*Example 7* draws on the same thinking as the previous two examples, but here explicitly refers to the upper bound and the misconception that choosing the upper bound will always result in the maximum answer. Consider the **language** that students use to approach this question. How readily can they explain the effect of a change on one value? Have you introduced terms like 'quotient', 'dividend' and 'divisor' to explore division, or 'difference'; 'subtrahend' and 'minuend' to explore subtraction? Does the use of such terminology support explanations, by allowing students to be specific about what they are referring to?
Choose appropriately the upper or lower bound in a calculation *Example 8:* *In a factory, a machine fills 1 kg bags of flour, correct to the nearest 10 g.* *a) If the machine can fill 30 bags per minute, what is the minimum amount of flour that might be weighed out in an hour?* *b) If 600 kg of flour is put into the machine, what is the minimum number of bags that might be produced?*	The **variation** in *Example 8* draws attention to the choice of calculation and how that might be affected by the numbers chosen. The context and the limits of accuracy are the same throughout, but the approach to the context is different. The intention is that students are not distracted by trying to understand a new context each time, and so can focus on what mathematical structures underpin the questions.

c) If 200 bags of flour have been produced, what is the minimum amount of flour that might have been used?	
Example 9: *A typical household oil tank has a capacity of 1 200 litres, correct to the nearest litre.* *A typical oil tanker lorry has a capacity of 20 000 litres, correct to the nearest 100 litres.* *A typical oil tanker can be filled or emptied at a rate of 80 litres per minute, correct to the nearest 100 ml.* *Write a question involving two of these facts where you need to use:* *a) both lower bounds* *b) both upper bounds* *c) one each of the upper and lower bounds.*	*Example 9* supports students' **deepening** understanding of how the structures of calculations need to be considered when choosing upper and lower bounds in calculations. Rather than completing the questions, they need to interpret the context in such a way as to require different combinations of upper and lower bounds. You could develop this further by asking them to generalise what kind of calculations might always (or never) be used in the answers to all three parts.

7.1.2.1 Understand the mathematical structures that underpin multiplication and division of numbers represented in standard form

Common difficulties and misconceptions
When calculating with standard form, it is common for students to convert to a more familiar format before calculating, and then convert the resulting value back to standard form. This is inefficient, so it is important to help students to recognise the power of ten within their calculation as a 'unit' which can be added and subtracted so long as the 'units' are the same. Even if a student stays with standard form throughout the calculation, it is common for the result of the calculation to need conversion to standard form (for example, $2 \times 10^5 \div 4 \times 10^3 = 0.5 \times 10^2$ needs to be adjusted to 5×10^1 to be in standard form) and this conversion can sometimes be omitted or carried out incorrectly. Key to working with standard form is understanding that it is a factorised representation. The associative rule allows these factors to be multiplied in a different order to that in which they are presented.

Students need to	Guidance, discussion points and prompts
Recognise that standard form represents a number as a product of two factors *Example 1:* *Complete the calculations.* *a)* $12 \times 2 \times 5 \times 10 = 12 \times$ __	When multiplying numbers represented in standard form, it is important that students understand that the factors can be written in any order while maintaining equality. The **variation** within *Example 1* gives an opportunity for students to experience using the associative law to identify powers of 10 and reorder multiplications so that they are written with a power of 10 as a factor. Part *b* of the

b) $4 \times 3 \times 2 \times 5 \times 10 = 12 \times$ ___ *c)* $4 \times 3 \times 0.2 \times 0.5 \times 0.1 = 12 \times$ ___	example builds on part *a* but the 12 is rewritten as 4×3; part *c* uses the same structure but with decimals. Students might want to calculate from left to right. What actions might a teacher take to draw attention to the benefits of identifying and working with the powers of 10? How would this change with part *c* and the use of decimals?
Example 2: *a)* *Do* 6×10^5 *and* 8×10^5 *have a common factor (other than 1)? Explain how you know.* *b)* *Do* 6×10^5 *and* 7×10^5 *have a common factor (other than 1)? Explain how you know.*	In *Example 2*, we are considering that standard form is simply a **representation** of a number. Here, the context of a common factor is used to revisit the idea that numbers written in standard form can be thought of as factor pairs. A teacher might introduce the task to a class by considering products such as $3 \times 5 \times 7$ and 3×8 and discussing whether these gave a common factor, before working with the less familiar context of powers of 10.
Example 3: *Multiply* 6.1 *by* 10^7*. Give your answer in standard form.*	This task gives an opportunity to discuss the different meanings of the **representation** 6.1×10^7, considering it both as an object and a process (otherwise known as a 'procept' – see the overview in '*7.2 Using structure to transform expressions'* for more detail). Predict what your students will do with this question. Will they write it in the normal form and then convert back to standard form? Do you think your students would answer differently if the calculation was offered as 10^7 multiplied by 6.1, or presented as $10^7 \times 6.1$? Try this task with a class and see how accurate your predictions were.

Apply the associative law to reorder multiplications and divisions of numbers represented in standard form to allow for efficient evaluation

Example 4:

Look at the calculation $7 \times 25 \times 3 \times 4$.

Matilda and Elinor work out the calculation in different ways.

Matilda's method:

$$7 \times 25 = 175$$
$$3 \times 4 = 12$$
$$\begin{array}{r} 175 \\ \times \quad 12 \\ \hline 350 \\ 1750 \\ \hline 2100 \end{array}$$

So $7 \times 25 \times 3 \times 4 = 2100$

Elinor's method

$$25 \times 4 = 100$$
$$7 \times 3 = 21$$
$$21 \times 100 = 2100$$

a) *Which method do you think is easier to work out?*

b) *Which method would you use, and why? It could be one of these two, or another method.*

In *Example 4*, students are invited to review two correct methods for evaluating a calculation. You could set the multiplication as a question for your class and use the students' methods to compare – showing Matilda and Elinor's methods only if they don't arise from their work.

This task draws attention to the efficiency of writing a product where one of the factors is a power of 10 and gives the opportunity for students to appreciate that using the associative law allows them to reorder the calculation to minimise the effort needed to calculate.

If you used this task with your class, would you give the calculation to evaluate without first showing Matilda and Elinor's methods, or would your starting point be to show the two given methods? Consider the benefits of showing or withholding the methods at the start of the task.

Example 5:

Look at the calculation $3 \times 10^4 \times 2 \times 10^3$.

Ruth and Sameen work out the calculation in different ways.

Ruth's method:

$$3 \times 10^4 \times 2 \times 10^3$$
$$3 \times 10^4 = 30\,000$$
$$2 \times 10^3 = 2000$$
$$30\,000 \times 2000 = 60\,000\,000$$
$$60\,000\,000 = 6 \times 10^7$$

So $3 \times 10^4 \times 2 \times 10^3 = 6 \times 10^7$

Example 5 follows the same pattern. Again, you could set the calculation as a question for your class first to compare methods.

This task models the use of the associative law to rearrange a multiplication in order to calculate more efficiently. Both methods reach a correct solution but here the **variation** is in comparing the methods, and drawing attention to the structures that they rely on, rather than reaching the right answer.

Sameen's method includes a line of working where the calculation has been re-written as $3 \times 2 \times 10^4 \times 10^3$ to make the associative law explicit. A student who is fluent in multiplying numbers in standard form might not show this step.

Sameen's method: $3 \times 10^4 \times 2 \times 10^3$ $= 3 \times 2 \times 10^4 \times 10^3$ $= 6 \times 10^7$ *Which method would you use? Which method do you think is easier to work out?*	Discuss this with your colleagues. Do you think that it has a use in this example, or would you omit that step when working with your students? There are arguments for and against its inclusion, which may depend on how well embedded the associative law is in your curriculum.
Example 6: *Which is greater?* $(1.9 \times 10^7) \div (2 \times 10^3)$ *or* $(1.9 \div 2) \times (10^7 \div 10^3)$	The equivalent calculations in *Example 6* are written in two ways. The intention of the **variation** is that students discuss the calculations, giving the teacher the opportunity to identify any misconceptions or gaps in knowledge. The prompt, *'Which is greater?*', draws attention away from evaluating the two calculations and towards considering their structure. As with the previous two examples, draw students' attention to the way that the associative law can be used to reorder a calculation to make it easier to work with. Consider the **representations** used in these calculations. Students can find it more challenging to work with the associative law in the context of division. They might not immediately recognise that these two calculations are equivalent as there are different numbers of division/multiplication symbols in each calculation. They may find it easier to recognise the equivalence when the divisions are expressed as fractions.
Understand and write equivalent factorisations for numbers represented in standard form *Example 7:* *4 710 written in standard form is:* 4.71×10^3 *Complete calculations a to f.* *a)* $4.71 \times 10^4 = 4\,710 \times$ __ *b)* $47.1 \times 10^3 = 4\,710 \times$ __ *c)* $47.1 \times 10^4 = 4\,710 \times$ __ *d)* $4.71 \times 10^3 \times 10 = 47.1 \times 10^3 = 4.71 \times 10^{[\_]}$ *e)* $4.71 \times 10^3 \times 1000 = 4710 \times 10^3 = 4.71 \times 10^{[\_]}$ *f)* $4.71 \times 10^3 \div 100 = 0.0471 \times 10^3 = 4.71 \times 10^{[\_]}$	*Example 7* is in two sections with planned **variation**. These should be tackled as two distinct, but related, tasks. Calculations *a*, *b* and *c* draw attention to the different ways that a multiplication by 10 can be represented in a number written in standard form. When students have worked on parts *a*, *b* and *c*, ask, '*What's the same and what's different?'* or '*Which of these three is the odd one out?*', to provoke discussion about the different representations. Calculations *d*, *e* and *f* then consider these different representations and where the impact of the multiplication of division can be shown. In each of parts *d*, *e* and *f*, the impact of the calculation on 4.71×10^3 is first shown on the value of 4.71, with the 10^3 staying constant. Students are asked to show the equivalent calculation while keeping 4.71 constant. The intention is to draw attention to the way that a multiplication or division can be distributed across the different terms, and the way that one factor can be compensated as the other changes. This is important when multiplying and dividing with numbers using standard form **representation** as the outcome of the calculation may need to be adjusted so that it is in the form $A \times 10^n$ where $1 \leq n < 10$. For example, $4 \times 10^7 \div 5 \times 10^3 = 0.8 \times 10^4$, which needs to be rewritten as 8×10^3 to be in the correct form.

Example 8: *Complete the following calculations, giving your answers in standard form.* *a)* $3 \times 10^6 \times 2 \times 10^5$ *b)* $5 \times 10^6 \times 3 \times 10^5$ *c)* $1 \times 10^8 \times 9 \times 10^2$ *d)* $3 \times 10^2 \times 3 \times 10^{10}$ *Valbona answers parts a, c and d correctly, but makes a mistake on part b.* *What mistake do you think Valbona might have made on part b?*	The **variation** in *Example 8* has been planned so that, when evaluating part b, the product is 15×10^{11} which then needs to be rewritten as 1.5×10^{12} so that it is given in standard form. All of the other calculations in the example can be evaluated without this second step, which draws attention to the one that is different. The intention is **deepening** students' understanding of standard form, particularly to be aware that there are situations where an adjustment is necessary. Asking students to write pairs of questions – one where the product does need adjusting and one where it doesn't – would draw their attention to the structures that underpin this, that for the calculation $A \times 10^n \times B \times 10^n$, the product $1 \leq AB < 10$.

7.1.2.2 Understand the mathematical structures that underpin addition and subtraction of numbers represented in standard form

Common difficulties and misconceptions

Working additively with numbers presented in standard form requires a deep understanding of place value and unitising. Students need to appreciate that numbers must be of the same magnitude to allow the operation to be carried out simply. Draw parallels between adding familiar numbers, such as 3 hundreds and 4 hundreds, or 3.6 millions and 2.1 millions. Known as 'unitising', this is the idea that quantities of the same 'unit' can be easily added (see Key Stage 3 '*2.1 Arithmetic procedures*', particularly 2.1.1 and 2.1.3).

Students might want to rewrite the numbers in standard form in their usual, base-ten format in order to calculate, and then convert the resulting value back to standard form. They should understand that, by ensuring that the two values are written with the same power of ten, they can be added or subtracted without this conversion.

Students need to	Guidance, discussion points and prompts
Understand that numbers of the same magnitude can be easily added and subtracted *Example 1:* *How many thousands is each of the following?* *a) Two thousand added to four thousand.* *b) Nineteen thousand subtracted from forty thousand.* *c) One thousand eight hundred added to five thousand.* *d) Two point seven thousand added to three hundred.*	*Example 1* uses the **language** of the place-value system to draw attention to the way that the same units can be easily added and subtracted (known as unitising). Stressing the thousands when reading the question, and encouraging students to do the same, will increase this focus. In part *d*, students might be encouraged to read the question in 'thousands' and then rephrase it to 'hundreds' to draw attention to the ways in which numbers can be manipulated to be of the same unit, facilitating addition.

Example 2:

Surinder and Catriona each calculate $43 \times 1.9 + 5.7 \times 19$.

Surinder's method:

```
  43        57
× 19      × 19
 387       513
 430       570
 817      1083
```

So $43 \times 1{\cdot}9 = 81{\cdot}7$
and $5{\cdot}7 \times 19 = 108{\cdot}3$

```
  108·3
+  81·7
  190·0
```

Catriona's method:

$$\begin{aligned} & 43 \times 1{\cdot}9 + 5{\cdot}7 \times 19 \\ = \ & 43 \times 1{\cdot}9 + 57 \times 1{\cdot}9 \\ = \ & 100 \times 1{\cdot}9 \\ = \ & 190 \end{aligned}$$

a) *Describe how each method works.*

b) *Which method would you use? Why?*

Here, the **variation** is not in the questions but in the solutions: two different methods to reach a calculation are shown. This allows for comparison between a correct, though maybe inefficient approach (Surinder's), and an approach that might be considered more structural (Catriona's). This example offers a context for students to understand unitising, since Catriona's method relies on both addends being rewritten to have 1.9 as a factor, allowing for easy addition.

Recognise that numbers written in standard form can be rewritten to allow for ease of addition and subtraction

Example 3:

These numbers are in standard form. Rewrite them all in the form $A \times 10^4$.

a) $3.1 \times 10^5 =$ __ $\times 10^4$

b) $3.1 \times 10^{10} =$ __ $\times 10^4$

c) $7.1 \times 10^2 =$ __ $\times 10^4$

d) $5.01 \times 10^3 =$ __ $\times 10^4$

e) $6.8 \times 10^{-3} =$ __ $\times 10^4$

Example 3 is a tool for recognising standard form as one of many **representations** of number. Students might be used to rewriting the results of calculations so that they are in standard form (i.e. rewriting $A \times 10^n$ so that $1 \leq A < 10$). However, they may not fully understand that numbers in this form can be rewritten in many ways. Here, they rewrite the values so that the factor of 10^4 is apparent.

Consider the different strategies that students might use to rewrite each value. What are the benefits of an informal strategy (such as a student who offers the answer '10^4 *is ten times smaller than* 10^5 *so I need to make* 3.1 *ten times bigger*') versus a more consistent approach (such as decomposing the power of 10 each time)? For example:

$$3.1 \times 10^{10} = 3.1 \times 10^6 \times 10^4 = 3\,100\,000 \times 10^4$$

Example 4:

Jane and Nicola calculate $4.3 \times 10^5 + 5.7 \times 10^6$.

Jane's method:

$$4.3 \times 10^5 + 5.7 \times 10^6$$
$$= 4.3 \times 10^5 + 57 \times 10^5$$
$$= 61.3 \times 10^5$$
$$= 6.13 \times 10^6$$

Nicola's method:

$$4.3 \times 10^5 + 5.7 \times 10^6$$
$$= 430\,000 + 5\,700\,000$$

$$\begin{array}{r} 5\,700\,000 \\ +\quad 430\,000 \\ \hline 6\,130\,000 \\ {\scriptstyle 1} \end{array}$$

$$6\,130\,000 = 6.13 \times 10^6$$

Describe how each method works. Which method would you use and why?

As in *Example 2*, two different methods are offered with the intention of further **deepening** students' understanding of how manipulating the form of a number can open up different possiblities for calculating. Comparing the methods might lead to conclusions around their efficiency.

Example 4 gives a context where unitising is an underpinning idea. Consider with colleagues what other mathematical contexts are underpinned by unitising, such as collecting like terms. Would it be useful for students to be explicitly taught about unitising? You can refer to the following NCETM materials to support your discussion:

- Checkpoints 14-16 from the '*Expressions and equations*' deck.
- Core concept document '*1.4 Simplifying and manipulating expressions, equations and formulae'*.
- Core concept document *'2.1 Arithmetic procedures'.*

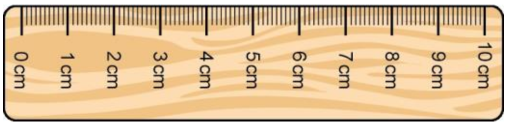
7.1.3.1 Understand the infinite nature of recurring decimals

Common difficulties and misconceptions

Using the formal written method for division can help to give further insight into the repeating and unending nature of a recurring decimal. To develop this insight, students need to understand fractions as a quotient and not only as a part–whole model. They also need to understand the idea of a procept: that the fraction is both an operation and the result of that operation. These ideas are discussed in the Key Stage 3 PD materials, particularly '*1.3 Ordering and comparing'.*

Some students may not understand that, when writing the result of a division, the fraction and the decimal representation of the quotient are equivalent. They are different ways of writing the same value. The use of a number line, and the understanding that the fraction and decimal occupy the same position when placed on that number line, might help to develop this awareness.

An additional challenge is the understanding of infinity. Students need to understand that a recurring decimal has no end, and this can cause a challenge when working with values such as 0.9 recurring. Many students will believe that this is less than one rather than an alternate way of writing the same value. Again, zooming in on these values on an interactive number line may help. See also *Example 4* in key idea 7.1.1.3, for further discussion of how to deal with the concept of infinitely recurring decimals in calculations.

Students need to	Guidance, discussion points and prompts
Understand that decimals and fractions are both representations of the same value *Example 1:* *Mrs Cardy asks her class to draw a 10 cm line using a ruler, and then split it into 3 equal parts.*  *Here are some of their responses.* *Asma says, 'My line is 3.3 cm long.'* *Benji says, 'This is impossible! 10 doesn't divide by 3.'* *Cara says, 'My line is* $\mathbf{3\frac{1}{3}}$ *cm long.'* *Declan says, 'My line is 3.33333 cm long.'* *Eric says, 'My line is* $\mathbf{3.\dot{3}}$ *cm long.'* a) *Who do you most agree with? Why?* b) *Who do you least agree with? Why?*	*Example 1* uses a simple context to explore students' understanding of different **representations** of a number that cannot be expressed as a terminating decimal. The prompts are ambiguous to promote discussion, so students may have different opinions about who they most or least agree with. The intention is to ensure students have an awareness of recurring decimals and the most accurate way to represent them. For example, although Declan's answer is more accurate than Asma's, as it has more decimal places, it is still an approximation. Students may be happier working with decimals, and so suggest that Eric is more accurate than Cara. This is worth addressing so that students appreciate that fractions are accurate representations of all numbers, including non-terminating decimals. Note that the recurring symbol is easy to overlook – so it is worth clarifying with students what is the same and what is different about Asma and Eric's responses. Consider students' experiences of fractions and decimals in classroom activities. Do they prefer to express answers in decimal rather than fraction form? Why or why not? What might be the implications for later work on expressing answers in terms of π, or in surd form? See the core concept document '*7.2 Using structure to transform and evaluate expressions'* for further discussion of this.
Recognise that a fraction represents a division and the outcome of that division can be represented by an equivalent decimal *Example 2:* $\frac{7}{16} = 0.4375$ a) *Use this fact to fill in the gaps:* $\_\_\_\_ \div \_\_\_\_ = 0.4375$ b) *Write a word problem that uses this calculation.* c) *Write another word problem that uses the calculation. What is the same and different about your questions?* d) *Think about the format of the solutions for your word problems. Is a decimal or fraction more appropriate, or does it not matter?*	*Example 2* makes explicit the link between a fraction, the equivalent fraction, and the resulting decimal. The use of a context to connect to these calculations offers scope to consider the **language** we use around division, fractions and decimals. This may make more apparent students' understanding of the equivalence between the different representations. Consider, when students 'read' the equation aloud, what their interpretation of the equals symbol is. Do they have an understanding of it meaning 'is equivalent to', or do they see it as meaning 'results in'? Students often consider the decimal to be the 'real' answer, without appreciating that the fraction is also a valid representation of the same number. Asking students to tell a 'story' for the calculation (for example, '*Seven metres of ribbon is cut into 16 pieces. How long is each piece?')* offers a further opportunity to hear students' understanding of division. For further **deepening** of students' understanding, ask them to write more division calculations that also give 0.4375 as an answer. This may also make more apparent the connection between equivalent fractions and the resulting divisions.

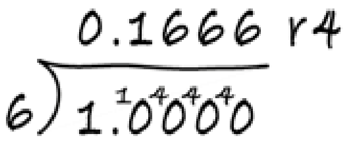
Understand that the formal written method for division can be used to write a fraction as a terminating decimal *Example 3:* *Look at these calculations.* $4\overline{)\,1\;2}$ $4\overline{)\,1\;3}$ $4\overline{)\,1\;\;\;\;}$ *For each one:* *a) Decide how you would read the calculation out loud.* *b) Decide what you expect the answer to be before you calculate.* *c) Calculate the answer using the formal written method for division.*	*Example 3* uses the standard division algorithm to find equivalent decimals. The numbers have been chosen so that the calculations are simple, and would not ordinarily be most efficiently solved using this method. This allows the focus to be on interpretation of the calculation and the result, rather than on 'getting the answer'. Make explicit the calculation being carried out by asking students to read each one out loud, paying attention to the **language** they use. You may find that students read the calculations from left to right (for example, incorrectly reading the first calculation as 'four divided by twelve') or that they habitually read the largest value as the dividend (for example, correctly reading the second calculation as 'thirteen divided by four' but then reading the final calculation as 'four divided by one'). The **variation** is such that very little changes between each calculation but the second and third calculations both have a non-integer answer, and so require students to use place-holder zeros to carry out the algorithm. Situations where the dividend is greater than the divisor may be more familiar to students, and so they are able to complete the process. Calculations where the divisor is the lesser value are less familiar and so less accesssible for students. Understanding that additional zeros can be added is fundamental to this key idea and is explored further in *Example 4* below.
Example 4: *a) Use the formal written method for division to calculate each of the following.* $8\overline{)\,5\;0\;0}$ $8\overline{)\,5\;\;\;\;\;\;}$ $5\overline{)\,8\;\;\;\;\;\;}$ *b) Use your answers to write equivalent decimals for* $\frac{8}{5}$ *and* $\frac{5}{8}$.	The **variation** in this task focuses on the way that additional zeros can be written after the decimal point, without changing the value of the dividend. Writing such equivalent dividends means that it is then possible for the formal written method for division to reach a final value. The first calculation offers a context in which some of the required zeros are visible. Connecting this to the second and third calculations (maybe using prompts, such as, '*What's the same and what's different?*') may help students to appreciate the underlying structure.

Understand that the formal written method for division can be used to write a fraction as a recurring decimal

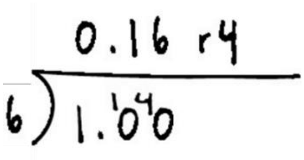
Example 5:

Akram, Brenton and Cameron use the formal written method for division to write $\frac{1}{6}$ as a decimal.

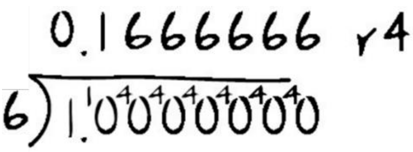
Akram writes:



Brenton writes:



Cameron writes:



a) *Explain why none of them has completed their calculation correctly.*

b) *What does the r4 mean in each calculation?*

Example 5 considers the recurring nature of $\frac{1}{6}$ and the **language** of remainders in division. The notation used is ambiguous, to generate a discussion. Students need to understand why the notion of a 'remainder' is inconsistent here and that different notation is necessary to show that a decimal expansion of $\frac{1}{6}$ continues with an infinite number of sixes. Probe the meaning of the 'remainder' with questions such as, '*Four what's are remaining in each person's answer?*'

Do you think that *Example 5* gives an effective opportunity to raise students' awareness of the infinite nature of the decimal expansion? If not, how might you raise this awareness with your students?

Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 7 Using and applying numerical structure* can be found at

https://www.ncetm.org.uk/media/y01pnrgg/ncetm_ks4_cc_7_solutions.pdf