Secondary Magazine 123

National Centre for Excellence in the Teaching of Mathematics

At last the summer break is here, and we here at NCETM Central are packing our suitcases: flip flops? Check. Sunscreen? Check. Holiday reading? Hmm … should we again lug the 2 kgs or so of *A Suitable Boy* and *The Goldfinch* with us on our travels, or just accept that once again we're not going to read these magna opera, and instead relax on our beach towels and gently ponder the ideas and suggestions in this and the next issues of the secondary magazine? That way we'll be ready for the shock to the system that will be 1 September, and we'll have more room in our cases for souvenirs and the inevitable "why did we bring this home with us?" local produce! As always, your comments and feedback are very welcome, either at the bottom of the page, by email to info@ncetm.org.uk, or @NCETMsecondary on Twitter.

Contents

Image credit

Heads Up

Whilst not wanting to define this article as mathematical gossip, it does come close! We've brought together news and current mathematical affairs, all in one place. We do hope it will interest you.

New versions of the Sample Assessment Materials for the new GCSE have now been approved by Ofqual and each of the awarding bodies has now released the questions:

- AQA
- Edexcel
- Eduqas
- OCR.

This year is the 200th anniversary of the birth of George Boole, the mathematician who formalised the ideas we now call Boolean Logic - which were discussed, pleasantly surprisingly, on Radio 4's Today programme, 2h 53:40 in. His wife Mary was also a mathematician; what must their dinner table talk have been like?!

In addition, the University of Lincoln is hosting a free public exhibition, The Life and Legacy of George Boole, as part of a programme of events celebrating Boole's life. The exhibition runs 13 July - 11 September at the University's Library, before moving to Lincoln Cathedral in October.

Another exhibition, The Amazing World of M. C. Escher, is running at the Scottish National Gallery of Modern Art this summer, with an accompanying programme of lectures. Let's hope it's on the ground floor, and not up, and round, and down, and round, and up, and round, and down, and round, and up…a flight of stairs! It runs until 27 September.

Would you like to visit other countries to see for yourself how their education systems work? Are you in a position to disseminate the new knowledge and examples of best practice that you will observe, for the wider benefit of your community and the UK? Then you could apply for a Churchill Fellowship. The Winston Churchill Memorial Trust (WCMT) awards Travelling Fellowships to British citizens to travel overseas and to bring back knowledge and best practice for the benefit of others in their UK professions and communities. Fellows receive a travel grant to cover return and internal travel, daily living and insurance within the countries visited. Applications for travel in 2016 are now open, but must be made by 5pm on 22 September 2015.

And finally, big NCETM congratulations to Colin Hegarty, who teaches at a north London secondary school. Colin has won one of Richard Branson's £50 000 *Pitch to Rich* prizes, to develop his maths tuition website and make it free for all children. You can read more here.

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Building Bridges

Hey Diddle Diddle the median's the middle.
You add and divide for the mean.
The mode is the one that you see the most,
And the range is the difference between.

Moving from the process skills of calculating these summary statistics to deeper contextual understanding of which to use, and when, and why, is often tricky for pupils. The problem in part is one of memory: quite often you will hear a pupil say "medium" instead of "median" and then muddle the other two. This alliterative nightmare continues right up to the GCSE exam for some. One response is to teach each element separately and as far apart in the curriculum as you can manage, to try to prevent pupils from muddling the meaning of mean, median and mode. This strategy is explored in more detail in this blog by Bruno Reddy.

But keeping the topics apart in the curriculum and teaching the processes is not, of course, "job done". When pupils can calculate mean, median and mode, do they have the full understanding of which one to use and how to interpret the value(s) they calculate? Could they make a decision about which is the best statistic to apply, given the scenario, and why? This diagnostic question about the use of the mean average could easily be adapted to cover the other three summary statistics:

Ten pupils take part in some races on Sports Day, and the following times are recorded. For each race, do you think that the mean average of the times would give a useful summary of the ten individual times? Explain your decision.

- *Time to run 100m (seconds): 23, 21, 21, 20, 21, 22, 24, 23, 22, 20.*
- *Time to run 100m holding an egg and spoon (seconds): 45, 47, 49, 43, 44, 46, 78, 46, 44, 48.*
- *Time to run 100m in a three-legged race (seconds): 50, 83, 79, 48, 53, 52, 85, 81, 49, 84.*

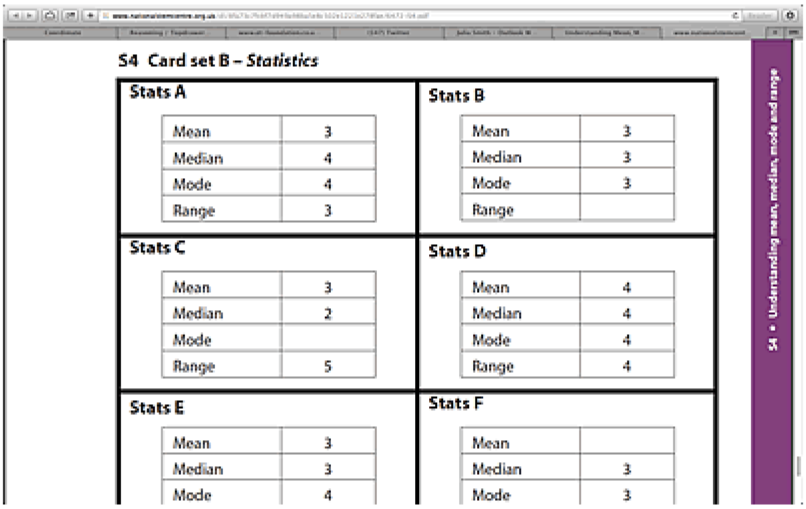
In *Eight Effective Principles of Mathematics Teaching*, Malcolm Swan demonstrates the importance, and impact, of questioning. A powerful technique is to turn the questions around, from simple closed question to "Show me" open questions: rather than ask "What is the mean, median and mode of 5, 4, 5, 6, 5?", ask instead "If the mean of a set of numbers is 5 what might the numbers be?" and then "If the mode is 4 and the mean is still 5 what might the numbers be?". Or, "If the mean, median and mode of a set of six numbers is 6, what might the numbers be?"

"Always, sometimes, never" statements challenge and deepen pupils' thinking and reasoning. The examples here are from NRICH:

1. *The mean, median and mode of a set of numbers can't all be the same.*
2. *The mean cannot be less than both the median and the mode.*
3. *Half of the students taking a test will score less than the average mark.*
4. *Nobody scores higher than the average mark in a test.*
5. *In a game where you can only score an even number of points (0, 2, 10 or 50), the average score over a series of games must be an even number.*

Alongside ensuring that pupils can calculate summary statistics fluently, you can deepen their understanding using some of the following resources:

- the Standards Unit [Mostly Statistics](#) materials S4 *Understanding Mean, Median, Mode and Range*


S4 Card set B – *Statistics*

Stats A

Mean	3
Median	4
Mode	4
Range	3

Stats B

Mean	3
Median	3
Mode	3
Range	

Stats C

Mean	3
Median	2
Mode	
Range	5

Stats D

Mean	4
Median	4
Mode	4
Range	4

Stats E

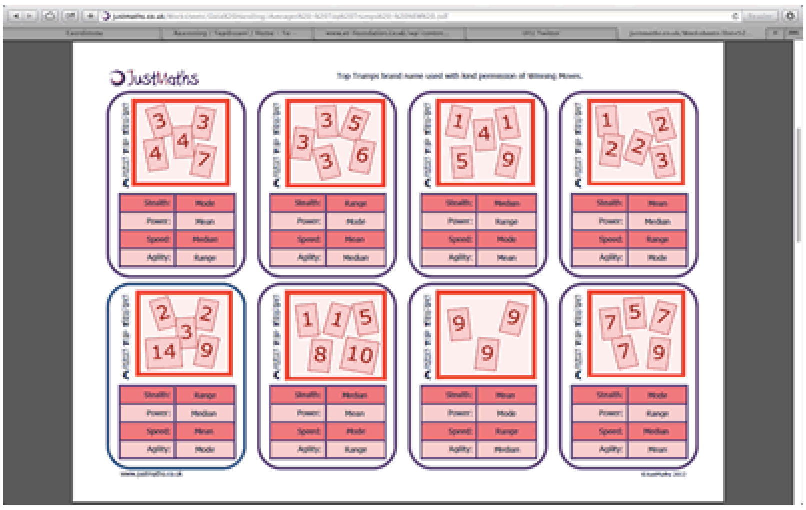
Mean	3
Median	3
Mode	4

Stats F

Mean	
Median	3
Mode	3

S4 • Understanding mean, median, mode and range

- the follow-on unit is [Interpreting Bar charts, Pie charts, Box and whisker plots](#), to help pupils develop further the connections between raw data and its visual representation and analysis;
- [Learn and WinAtSchool](#) offers a range of resources as well as an annual competition. The activity [The Averages](#) is an enjoyable challenge;
- within the Durham Maths Mysteries there is the excellent [Ratio and Proportion Mystery](#), which draws on knowledge of the mean average;
- and as the end of term approaches…everyone loves a game of [Top Trumps](#)!



You can find previous *Building Bridges* features [here](#).

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Sixth Sense

If you're anything like me, you're looking forward to a few weeks away from school and probably don't want to think too much about September just yet. However, at this time of year – and better now than at midnight on 31 August! – it is worth reflecting on what has gone well and where there is room for improvement in one's teaching. If you're a subject lead then this reflection might well feed into a re-drafting of your schemes of work: at KCLMS, as at most schools, these are very much working documents which are constantly being tweaked and (we hope!) improved.

Take the beginning of teaching AS Mechanics as an example. If, like us, you teach this to Year 12, this is probably the first time students will encounter "Applied Mathematics" and there is certainly a conceptual jump from "mathematicians learn things which are true" (Pythagoras' Theorem, how to solve quadratic equations, the uniqueness of prime factorisation) to "mathematicians use models to make predictions that will be compared to observations made in the real world" (stock prices, rates of decay, punctuality of trains). Even if you leave Mechanics until Year 13, I'd argue that much of the following still applies.

In order to reach the point where students can tackle confidently the typical disaster-movie-scenario exam question in which "a train and N trucks are driving up a hill when the coupling breaks", there is substantial conceptual understanding and procedural fluency that they need to develop: the forces model, Newton's Laws, components, the meaning of the assumptions regarding lightness and inextensibility, the constant acceleration formulae, etc. Each of these ideas, I suggest, needs approaching separately and pulling apart in some detail before they will be well-prepared to attempt this sort of question.

At KCLMS, we start by focusing on the principles of Mechanics in one dimension, and we leave any discussion of components until the spring term in Year 12. This gives the Core teacher time to review, revise, and improve students' trigonometric fluency before they need these skills in Mechanics lessons. You can go a long way in one dimension (pun fully intended!): even the Further Mathematicians study 1D impulse, momentum, moments and work done before they start to use components. I know that this isn't the standard approach – think of all the textbooks that start with "Chapter 1: Vectors" – but it's how I was encouraged to teach this material early in my career by my mentor who, in turn, picked it up from the Head of Maths at Westminster School. I've become ever more convinced by the sense of it each time I've taught the AS applied material.

Let's look in some detail at the first few lessons on our scheme: I'm hoping to convince you that a bit of showmanship and a high level of pedantry at the start of the course will enable students to solve difficult problems with confidence later on.

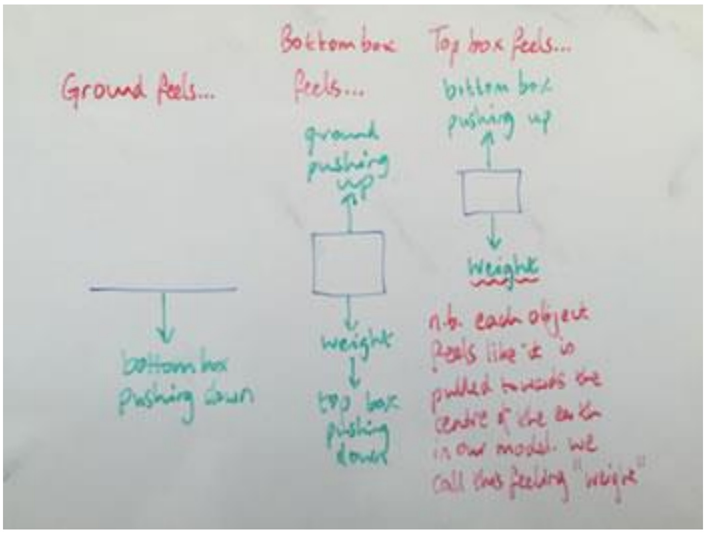
Lesson 1

Objective: to identify the number of separate objects in a system, and to describe (without formal labelling) the forces acting on each object.

Teaching: I will start this lesson by dropping a textbook, and asking "what just happened?" When the students say "it fell", I put it on a table and we discuss why it no longer falls. At this stage, I discourage the use (often misuse!) of formal language that they have picked up elsewhere: rather than using "tension", "reaction", etc., I will ask students to describe everything that each object "feels", in plain English for now. Most students talk about weight, and they have to talk about this to realise that this is something that the object experiences in and of itself: the **table** doesn't "feel" the weight of the **book**, but it does "feel" the book pushing down on it.

Example: Two boxes sit on the ground, one on top of the other. Draw each object separately and add the forces that each object "feels" on your diagram.

Solution on board [click image to enlarge]**:**



(the picture of the ground is quite uncluttered, but be prepared for some good questions from students here – does the ground have weight?; shouldn't there be something below the ground pushing up?; shouldn't we draw another picture of whatever this is? Ultimately I argue that we have to stop "zooming out" somewhere and the three objects shown above are the only three mentioned in the question!).

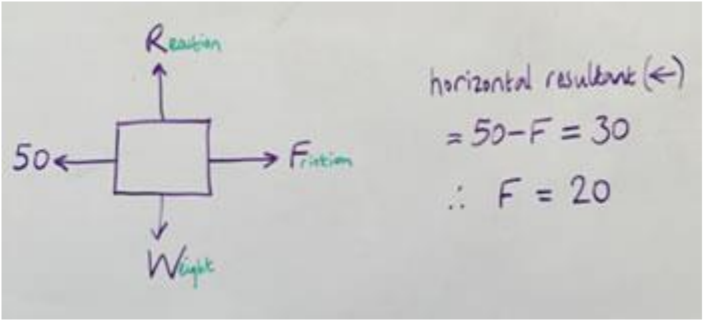
Lesson 2

Objective: to draw force diagrams, now with more formal labels, and to introduce the concept of a resultant force.

Teaching: There aren't new ideas here: as mentioned above, the students almost certainly used words like "tension", "reaction", "friction" in lesson 1 and I almost certainly said "I don't know what you mean by that", so in this lesson we're going to define those words and then begin to use them appropriately. In the latter part of this lesson, there's time for a discussion of the resultant force (the single force that has the same dynamic effect as the original set of forces) – this is where we will start to see some numbers.

Example: A book sits on a rough table-top. The book is pushed horizontally by a force of 50N. Draw a diagram showing all of the forces acting on the book, and find the size of the frictional force if the resultant force acting horizontally is 30N.

Solution on board [click image to enlarge]**:**



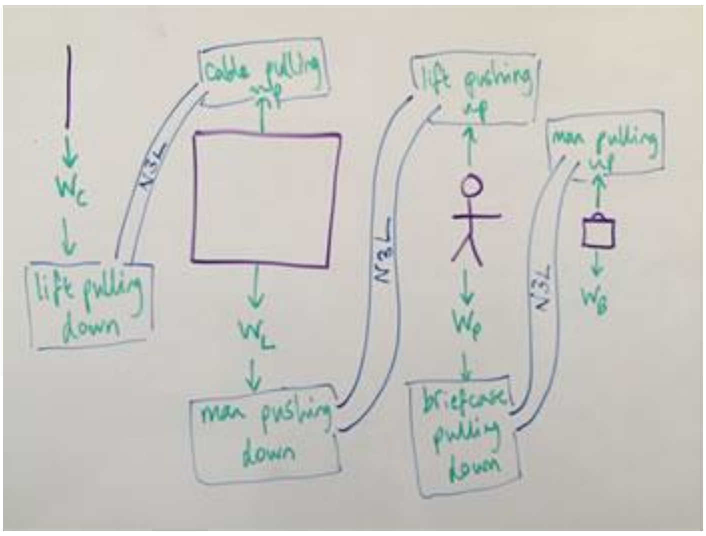
Lesson 3

Objective: to identify pairs of forces that Newton's 3rd Law says can be considered as being equal.

Teaching: I will tell my students that Newton was a genius (though surprisingly obsessed with alchemy and hocus-pocus as well!), but stress that he didn't prove that forces exist … for the very good reason that they don't. The key point I want to get across to my students is that forces are a model, and that we're trying to find a model that produces consistent explanations for what we have seen, and/or we're trying to use mathematics to make predictions that can then be compared with what we do see, in the real world. If our model is not good enough (and we will talk about what that means), we have to come up with something better. The impressive thing about Newton's model is that it has been "good enough" for over 300 years. Students may say that they've learned about his 3rd Law in Physics, and they'll chirp "action and reaction are equal and opposite". I will faux-wince and discuss with them the much better formulation that "if the forces which two objects cause to act on each other are modelled as equal and opposite, then the predictions that this model makes are not inconsistent with experimental observations".

Example: A person carrying a briefcase stands in a lift, which is held up by a cable. Draw force diagrams for each of these four objects, and then identify the pairs of forces that Newton's 3rd Law says can be modelled as equal and opposite.

Solution on board [click image to enlarge]**:**



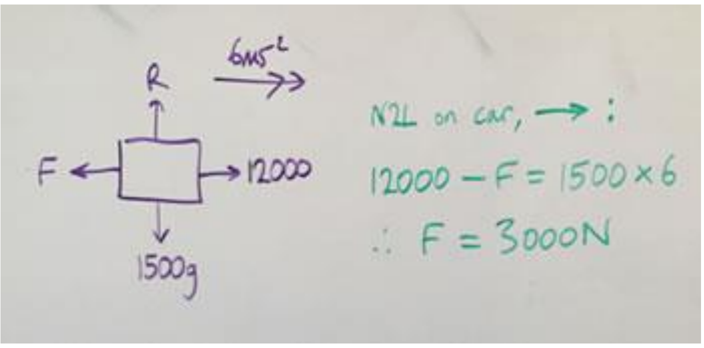
Lesson 4

Objective: to state Newton's 2nd Law in full, and to apply this in simple situations (including to calculate weight from mass).

Teaching: Starting from Newton's 1st Law, which in essence says that "if there's no resultant force acting on an object, then the object won't change its velocity", I will want my students to see that if there is a resultant force acting on an object, then that object will accelerate. I will explain to them that Newton's 2nd Law gives the detail: "we can model the resultant force acting on an object in a certain direction as being equal to the mass of the object multiplied by the acceleration in the same direction". I won't let my students shorten this to "F = ma": concision very rapidly leads to confusion.

Example: A car accelerates along a horizontal road. The car's engine generates a driving force of 12000N. Find the frictional force between the tyres and the road if the 1500kg car accelerates at $6ms^{-2}$.

Solution on board [click image to enlarge]**:**



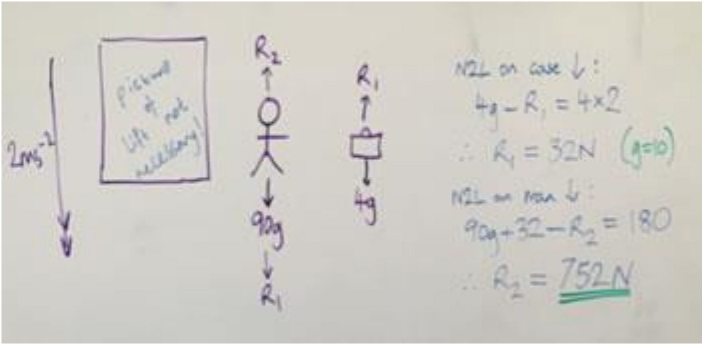
Lesson 5

Objective: to use Newton's 2nd Law to solve more complex one-dimensional problems

Teaching: What's important here is making it clear that Newton's 2nd Law acts on **AN** object, in **A** direction, so I will expect my students to specify these explicitly.

Example: Revisiting the man in the lift with the briefcase that we saw in an earlier lesson, find the reaction force between the man and the lift if his mass is 90kg, the mass of the briefcase is 4kg, and the lift accelerates downwards at $2ms^{-2}$.

Solution on board [click image to enlarge]**:**



I agree that we haven't yet done any questions as hard as those that come up on M1 papers, but these lessons should have laid strong foundations of the new ideas and the crucial concepts. We're now in a position to start looking at connected particles: three or four lessons on problems involving strings, pulleys, cars and caravans should ensure that these ideas now bed in.

Only several weeks later, once we've covered the constant acceleration formulae and used projectiles as a way into components, will we be ready to come back to our train and its dangerously connected trucks chugging uphill. With force diagrams and Newton's 2nd Law covered already, and trigonometry revised by the Core teacher, we won't be teaching deep concepts and relying on secure technical skills in one go: even with the most confident students, this is a good thing!

You can find previous *Sixth Sense* features here.

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From the Library

Why do some pupils find algebra so mystifying and difficult, but some just seem to "get it" and quickly develop both procedural fluency and conceptual understanding? Dietmar Küchemann carried out research in the 1970s examining how 14-year-olds interpreted the use of letters in algebraic contexts. His findings were reported and discussed in K. Hart, 1981, *Children's understanding of mathematics: 11-16 and Children's understanding of numerical variables* in *Mathematics in School*, September 1978.

Küchemann identified six different modes of understanding. Recognising the way we use these different interpretations might help us illuminate where our pupils' difficulties lie, and then to support them to overcome these. In the first three interpretations the letter is considered as a concrete entity (see The 'algebra as object' analogy: a view from school by Kate Colloff and Geoff Tennant, discussed in Issue 116) while the next three levels, on which we focus here, involve aspects of abstraction.

- A letter may be used as a specific unknown: "if $e + f = 8$, give an expression for $e + f + g$." This requires the letter to be understood to represent a specific but unknown value. Common incorrect answers to this question were $8g$, 12 and 9.
- The letter may be used as a generalised number where the letter could take a number of values: "what can you say about c if $c + d = 10$ and c is less than d?"
- Letters may be used as variables where there is a (second-order) relationship between the expressions: "which is larger, $2n$ or $n + 2$?". In his research, Küchemann was looking for answers of the form "it depends" with some evidence of substitution of, say, $n = 1$ and $n = 10$. A commonly seen answer was "$2n$ is bigger because multiplying makes things bigger". This is a very prevalent and hard-to-eradicate misconception (and is coupled with "division makes things smaller").

As mathematicians, we move adeptly between different interpretations of notation, suiting our reading of the variables to the context. But are our pupils as familiar, implicitly, with these interpretations? Are they accomplished in recognising which is applicable in different contexts? Küchemann used a number of carefully crafted questions to probe and clarify what pupils were thinking; these are available on the CSMS Tests area of the Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) website. You could use these items as an effective diagnostic tool for identifying the interpretations with which your pupils are comfortable and confident, and hence plan precisely the questions and activities which will first consolidate and then deepen their algebraic fluency, understanding and reasoning. ICCAMS, a major research project, was undertaken in 2008 to 2012 building on Küchemann's research. Information and trial teaching materials can be found on the ICCAMS website: well worth reading before planning your teaching of algebra next year.

You can find previous *From the Library* features here.

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It Stands to Reason

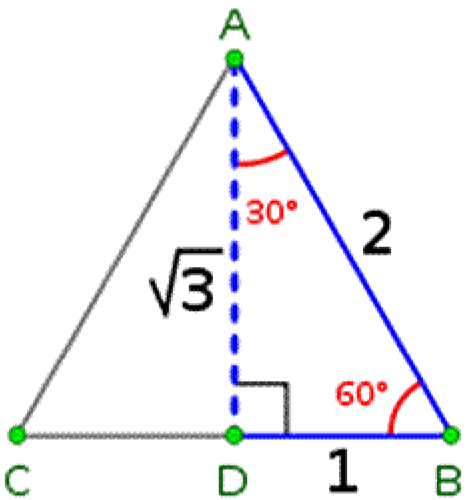
For the new GCSE pupils are expected to

- *know the exact values of sin θ and cos θ for θ = 0°, 30°, 45°, 60° and 90°*
- *know the exact value of tan θ for θ = 0°, 30°, 45° and 60°*

In the programmes of study the text is bold and underlined, so this is intended for the highest attaining pupils, for whom the factual recall and predictable exam questions that this material entails will not be inspiring. But taken as a context for inquiry, exploration and proof, this is a topic that can be rich and satisfying. @Letsgetmathing suggests starting with an activity in which pupils lean a 1m rule against a wall then measure the height of the triangle created, and hence calculate the sine of the angle for a range of different angles. They can estimate the values of sin θ in the special cases, and can also plot the values on a graph. This will suggest that sin 45° is **about** 0.7; they should ask if it's **exactly** 0.7, and if it's not, then what is it?

Start with an isosceles right-angled triangle (the short sides could be length 1, but better would be length *x*) on the board and ask what can be worked out: fingers crossed that "the length of the hypotenuse" and "the other two angles" are suggested quickly! The ratio opposite: hypotenuse (or, better, the scale factor from the length of the hypotenuse to the length of the opposite) is clearly 1/√2, as is the scale factor from the length of the hypotenuse to the length of the adjacent (or, the ratio adjacent: hypotenuse); thus we know exactly the values of sin 45° and cos 45°. That the scale factor from the length of the adjacent to the length of the opposite is 1 tells us the exact value of tan 45°.

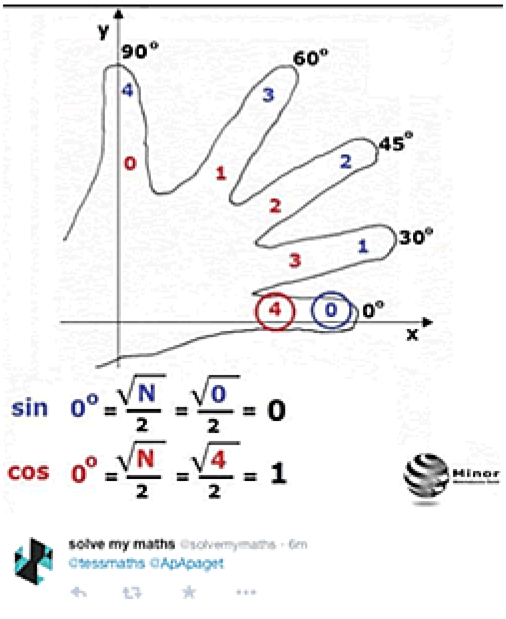
Once pupils have seen the idea that a well-chosen triangle can tell them exact values of trigonometric scale factors, the suggestion of using an equilateral triangle to determine, say, sin 60° should be quickly forthcoming from one pupil … and should be quickly challenged by another because it's not a right-angled triangle … and a third pupil will suggest the idea of bisecting it … and a fourth will use Pythagoras to work out the height of the triangle … well, that's the plan!



From this diagram the relevant scale factors come out naturally. It's well worth then asking pupils if they notice relationships between the surd expressions. They're likely to spot that cos 30° = sin 60° and vice versa, and this should be explained (by considering how the opposite and the adjacent swap, as it were, relative to the 30° and 60° angles), and then generalised to show that in any right angled triangle cos (90 – A) = sin A. Then they might consider, say, sin 30° ÷ cos 30°, and what they observe should be cross-checked with sin 60° ÷ cos 60° and sin 45° ÷ cos 45°; these then can be generalised to any right-angled triangle. It's a bit of a leap to consider, say, $\sin^2 30°$ ÷ $\cos^2 30°$; the hint to do so is in the square-root form of the surd expression. Again, what's found here should be cross-checked with 60° and 45°, and then could be generalised.

Evaluating sin 0° and sin 90° (and the cosine scale factor as well) is a good introduction to the idea of a limit: if pupils consider a very, very tall right-angled triangle, they should be able to see that the side opposite the nearly-90° angle is nearly the same length as the hypotenuse, and thus the sine scale factor is nearly 1 and it gets closer to that value as the angle gets larger – we say it tends towards the limit 1; similarly, the cosine scale factor of the nearly-90° angle will be nearly 0, and will carry on getting smaller and smaller, tending towards the limit 0. A similar limit argument should be developed to explain why we say that sin ° = 0 and cos ° = 1.

It's important that our pupils know why these trigonometric scale factors have these values – but they do also need to remember the values as well. There are lots of mnemonics for doing so; this one (from @solvemymaths) will help them have the knowledge at their fingertips (comedy drum roll optional):



This from Great Maths Teaching Ideas is neat:

or you might like Mr Barton's quirky left-hand finger methods.

Resourceaholic contains three good suggestions to challenge pupils' knowledge and retention of the scale factor values. The code breaking task is especially fun.

Making Maths: a Clinometer from NRICH is a lovely activity for exploring trigonometry, using real-life heights of wind turbines and trees.

Read previous *It Stands to Reason* features here.

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Eyes Down – a piece of cake?

This cake, on sale recently at the Royal Academy, is inviting to the palate and to the maths teacher as well. Look carefully towards the bottom of the picture and you'll see that a slice costs £4.95. Could your pupils come up with an estimated price for the whole cake, and how would they explain their reasoning?

Food for thought, indeed!

If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb) to us at info@ncetm.org.uk, with a note of where and when it was taken, and any comments on it you may have.

Read previous *Eyes Down* features here

Image credit