MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Multiplication and Division

2.7 Times tables: 2, 4 and 8, and the relationship between them

Teacher guide | Year 3

Teaching point 1:

Counting in multiples of four can be represented by the four times table. Adjacent multiples of four have a difference of four. Facts from the four times table can be used to solve multiplication and division problems with different structures.

Teaching point 2:

Products in the four times table are double the products in the two times table; products in the two times table are half of the products in the four times table.

Teaching point 3:

Counting in multiples of eight can be represented by the eight times table. Adjacent multiples of eight have a difference of eight. Facts from the eight times table can be used to solve multiplication and division problems with different structures.

Teaching point 4:

Products in the eight times table are double the products in the four times table; products in the four times table are half of the products in the eight times table. Products that are in the two, four and eight times tables share the same factors.

Teaching point 5:

Divisibility rules can be used to find out whether a given number is divisible (to give a whole number) by two, four or eight.

Overview of learning

In this segment children will:

- skip count in fours/eights and build up the four/eight times tables
- use four/eight times table facts to solve contextual and abstract multiplication problems, contextual quotitive and partitive division problems, and abstract division problems
- explore links between the two and four times tables, and between the four and eight times tables, applying their knowledge of doubling and halving
- explore links between products in the two, four and eight times tables
- derive and apply divisibility rules for two, four and eight.

The teaching points in this segment (with the exception of the final part of *Teaching point 4,* and *Teaching point 5*) follow a similar progression to that used when learning the five and ten times tables, and exploring the links between them (segment *2.4 Times tables: groups of 10 and of 5, and factors of 0 and 1*). By now, children should be gaining confidence in linking skip counting, grouping and multiplication to build up times tables. Through this segment, they will also be developing a greater sense of the connections between times tables.

Teachers are encouraged to continue building up the class multiplication chart as each times table is covered (introduced in segment *2.4*):

×	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6		12	15			24		30		
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12		24	30			48		60		
7	0	7	14		28	35			56		70		
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18		36	45			72		90		
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22		44	55			88		110		
12	0	12	24		48	60			96		120		

Key: new facts in this segment
previously learnt facts
relevant previously learnt facts (commutativity)

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Throughout this segment, teachers should use the chart, along with the relationships between the two, four and eight times tables, to support children in understanding that the four and eight times tables do not present 26 new facts to learn. The multiplication facts are either already known (through application of commutativity) or are related to known facts through doubling. However, in order for children to become fluent with these times tables, as well as using the connections already mentioned, they must become secure in their understanding of how the times tables are constructed and how each multiple is related to the adjacent multiples. As with all of the times tables, regular practice will be needed in order for children to become fluent, both in reciting the times tables (for example, *'One four is four, two fours are eight…'*) and with isolated multiplication facts (for example, *'I know that seven fours are twenty-eight.'*).

Since children have already been introduced to division (segment *2.6 Structures: quotitive and partitive division*) and calculation of quotients using multiplication facts, division is embedded in the times table practice steps in this segment (steps *1.10* and *3.10*). Teachers should ensure that contextual division practice encompasses both the quotitive and partitive structures of division. Similarly, children have already been introduced to the 'one equation, two interpretations' concept of commutativity (segment *2.5 Commutativity (part 2), doubling and halving*) where, for example, 7×4 can represent either seven groups of four, or four groups of seven. As such, practice steps *1.10* and *3.10* also include application of four/eight times table facts to solve problems about four/eight equal groups (distinct from problems about groups of four/eight that are considered earlier in the respective teaching points).

The idea of the 'non-concept' is used to strengthen children's understanding. For example, children are expected to identify representations in which the objects have *not* been grouped equally, or those that have *not* been used appropriately to represent a multiplication equation. In *Teaching point 4*, children explore numbers that are *not* multiples of two, four and eight; this serves both to strengthen children's ability to identify numbers that *are* multiples of two, four and eight, and to prepare them for formal work on remainders in segment *2.12 Division with remainders*.

An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast*. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.*

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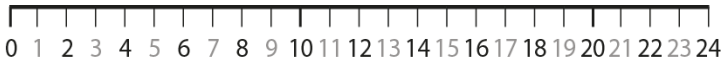
Teaching point 1:

Counting in multiples of four can be represented by the four times table. Adjacent multiples of four have a difference of four. Facts from the four times table can be used to solve multiplication and division problems with different structures.

Steps in learning

	Guidance	Representations
1:1	To begin this segment, briefly review skip counting in multiples of two between 0 and 24 (forwards and backwards), and the two times table. Use number lines, the Gattegno chart and contextual examples (such as pairs of shoes) for support. For more representations, see segment *2.3 Times tables: groups of 2 and commutativity (part 1)*. It is important that children understand the link between skip counting and the times tables before they begin work on the four times table. When skip counting, count in two ways: • *'Zero twos, one two, two twos, three twos…'* • *'Zero, two, four, six…'* (encourage children to keep a tally of the number of twos counted on their fingers) When reciting the two times table, refer to the factors in both orders, using the following language: • *'One two is two, two twos are four, three twos are six…'* (for $1 \times 2 = 2$, $2 \times 2 = 4$, $3 \times 2 = 6$…) • *'Two, once is two; two, twice is four; two, three times is six…'* (for $2 \times 1 = 2$, $2 \times 2 = 4$, $2 \times 3 = 6$)	Number line:  Gattegno chart: 1000 2000 3000 4000 5000 6000 7000 8000 9000 100 200 300 400 500 600 700 800 900 10 20 30 40 50 60 70 80 90 1 2 3 4 5 6 7 8 9 Contextual example: 

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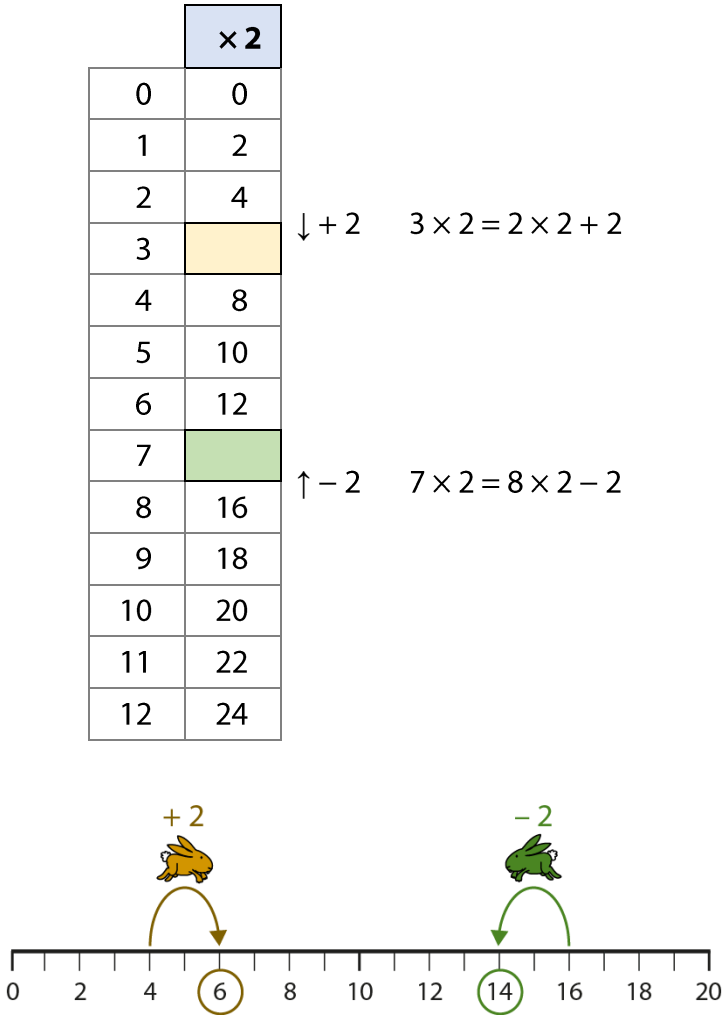
	Remind children that adjacent multiples of two have a difference of two. Note that, for now, we are focusing on a group size of two (not two equal groups), in preparation for working with groups of four/eight to build up the four/eight times tables. Later in the segment, children will apply their understanding of commutativity to use four/eight times table facts to answer multiplication questions about four/eight equal groups and about groups of four/eight, as well as both quotitive and partitive division problems.	Adjacent multiples of two have a difference of two: 
1:2	For the rest of this teaching point, we will build up and apply the four times table by following a similar sequence of learning to that used in: *2.3 Times tables: groups of 2 and commutativity (part 1)* *2.4 Times tables: groups of 10 and of 5, and factors of 0 and 1* Children will not have prior experience of skip counting in groups of four, and this is the first times table where they will not be building on existing fluency.	

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Begin by looking at some groups of four, linking enumerating objects in groups of four with counting in fours, and writing the associated multiplication equations (write two equations for each example, as shown opposite). Use contexts that children already associate with four; for example, wheels on a car, legs on a dog or sides of a square. Use four-value counters alongside each context to support the idea of unitising in fours, and use a number line with the multiples of four highlighted for skip-counting support.

For each example, ask children to describe what each number in the equation represents:

'What does the "5" represent?'
'The "5" represents the number of cars.'
'What does the "4" represent?'
'The "4" represents the number of wheels on each car.
'What does the "20" represent?'
'The "20" represents how many wheels there are altogether.'

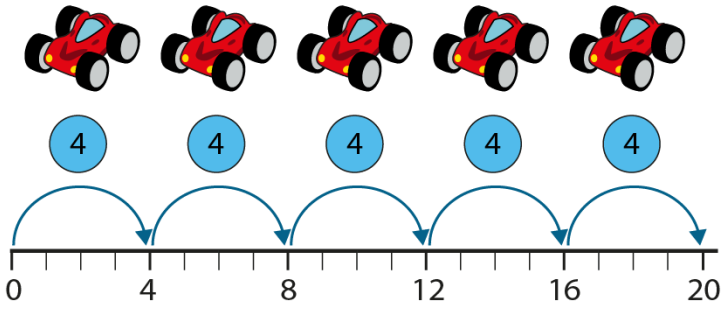
Remember, when describing a multiplication fact such as '$5 \times 4 = 20$' use the language '*five times four is equal to twenty*'. Avoid saying '*times by*' or '*multiplied by*'. For more on this, see segment *2.2 Structures: multiplication representing equal groups, Overview of learning*.

Also continue to use the language of factors and product to describe the multiplication equation:

'___ is a factor.'
'___ is a factor.'
'The product of ___ and ___ is ___.'
'___ is the product of ___ and ___.'

Work through several examples in this way, varying the representations used.

'How many wheels are there? Count in groups of four.'



- *'Four, eight, twelve, sixteen, twenty. There are twenty wheels.'*
- *'There are five groups of four; there are twenty altogether.'*
- *'There are four, five times; there are twenty altogether.'*

  $5 \times 4 = 20$ $4 \times 5 = 20$

- *'Five is a factor.'*
- *'Four is a factor.'*
- *'The product of five and four is twenty.'*
- *'Twenty is the product of five and four.'*

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1:3

Now consider zero fours. Remind children of the generalisation from segment *2.4 Times tables: groups of 10 and of 5, and factors of 0 and 1*: ***'When zero is a factor, the product is zero.'***

Remind children of how this generalisation applies to the two, five and ten times tables, and then apply it to the four times table; use a number line with backward jumps for support, and to verify that the generalisation holds true when the factors are zero and four.

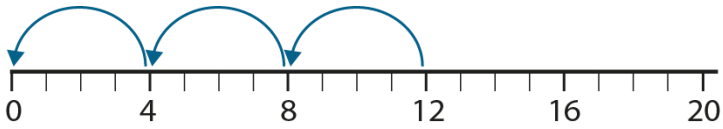
Reminder of the generalisation:

$0 \times 2 = 0$ $\quad$ $0 \times 5 = 0$ $\quad$ $0 \times 10 = 0$

$2 \times 0 = 0$ $\quad$ $5 \times 0 = 0$ $\quad$ $10 \times 0 = 0$

'When zero is a factor, the product is zero.'

Counting backwards to zero groups of four:

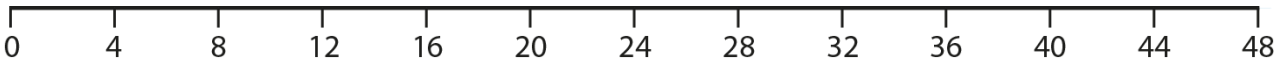


$3 \times 4 = 12$ $4 \times 3 = 12$	*'Three fours…'* or *'Four, three times…'*	*'Twelve…'*
$2 \times 4 = 8$ $4 \times 2 = 8$	*'two fours…'* or *'four, two times…'*	*'eight…'*
$1 \times 4 = 4$ $4 \times 1 = 4$	*'one four…'* or *'four, one time…'*	*'four…'*
$0 \times 4 = 0$ $4 \times 0 = 0$	*'zero fours'* or *'four, zero times.'*	*'zero.'*

1:4

Practise skip counting, forwards and backwards in fours between 0 and 48, regularly outside the main maths lesson, so that children begin to develop fluency with this counting sequence before moving onto the next step. Use familiar representations such as a number line and the Gattegno chart.

Number line:



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1:5

Now, using a familiar context, bring together the learning from steps *1:2–1:4*, working systematically to construct the four times table, beginning with zero fours.

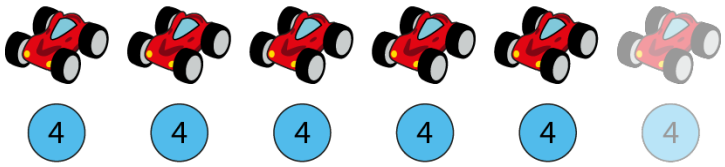
Use a ratio chart to record the number of groups and the product. As you complete the ratio chart, also write the multiplication equations; write *pairs* of equations for each times-table fact, using the following form of language:

- $3 \times 4 = 12$
  'Three groups of four is equal to twelve.'
  'Three times four is equal to twelve.'
- $4 \times 3 = 12$
  'Four, three times is equal to twelve.'
  'Four times three is equal to twelve.'

At each stage:

- encourage children to describe what each equation represents, for example:
  - *'There are three groups of four wheels.'*
  - *'There are twelve wheels altogether.'*
  - *'The product of three and four is twelve.'*
- then add another car, and work with children to complete the next column of the table, using their knowledge of what comes next in the counting sequence when skip counting in fours.

Building up the four times table:



$0 \times 4 = 0$	$4 \times 0 = 0$
$1 \times 4 = 4$	$4 \times 1 = 4$
$2 \times 4 = 8$	$4 \times 2 = 8$
$3 \times 4 = 12$	$4 \times 3 = 12$
$4 \times 4 = 16$	$4 \times 4 = 16$
$5 \times 4 = 20$	$4 \times 5 = 20$
$6 \times 4 = 24$	**$4 \times 6 = 24$**

Number of cars	Total number of wheels
0	0
1	4
2	8
3	12
4	16
5	20
6	**24**

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1:6

Once the ratio chart and full set of equations are complete, ask children questions, encouraging them to use the chart/equations for support, for example:

- *'If there are nine cars, how many wheels are there altogether?'*
- *'How many cars are there if there are twenty-eight wheels?'*
- *'If the product is forty, what are the factors?'*
- *'Why are eight times four and four times eight both equal to thirty-two?'*

Complete ratio chart and four times table:

Number of cars	Total number of wheels
0	0
1	4
2	8
3	12
4	16
5	20
6	24
7	28
8	32
9	36
10	40
11	44
12	48

$0 \times 4 = 0$	$4 \times 0 = 0$
$1 \times 4 = 4$	$4 \times 1 = 4$
$2 \times 4 = 8$	$4 \times 2 = 8$
$3 \times 4 = 12$	$4 \times 3 = 12$
$4 \times 4 = 16$	$4 \times 4 = 16$
$5 \times 4 = 20$	$4 \times 5 = 20$
$6 \times 4 = 24$	$4 \times 6 = 24$
$7 \times 4 = 28$	$4 \times 7 = 28$
$8 \times 4 = 32$	$4 \times 8 = 32$
$9 \times 4 = 36$	$4 \times 9 = 36$
$10 \times 4 = 40$	$4 \times 10 = 40$
$11 \times 4 = 44$	$4 \times 11 = 44$
$12 \times 4 = 48$	$4 \times 12 = 48$

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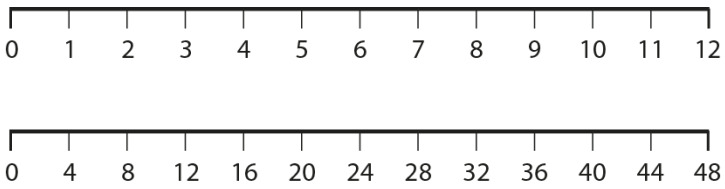
1:7	Now practise chanting the four times table, with the written times table for support, using a variety of representations, including: • stacked number lines (as shown opposite) • the Gattegno chart • concrete representations • pictorial representations. Use the following language: • *'One group of four is equal to four.'* *'Two groups of four is equal to eight…'* • *'One times four is equal to four.'* *'Two times four is equal to eight…'* then shortening to *'One four is four, two fours are eight…'* and • *'Four, one time is equal to four…'* *'Four, two times is equal to eight…'* • *'Four times one is equal to four…'* *'Four times two is equal to eight…'* Regular practice should be undertaken, including outside the main maths lesson, until children are fluent.	

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1:8

At this point, provide practice, including:

- completing/writing multiplication equations for contextual examples
- drawing/making contextual representations to match multiplication equations
- missing-number sequences and problems
- true/false style questions
- word problems, including measures contexts, for example:
  - *'What is the product of "8" and "4"?'*
  - *'Alvaro buys nine boxes of chocolates. Each box costs £4. How much does he spend altogether?'*
  - *'There are eleven children. Each child has four counters. How many counters is this altogether?'*
  - *'There are four seasons in each year. How many seasons are there in five years?'*

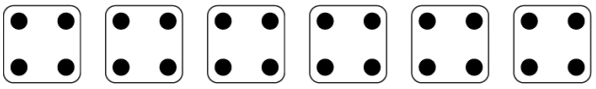
  Children should write a multiplication equation for each problem, rather than simply writing the product.

For word problems, ensure that some examples give four as the first piece of information, while others give it second (compare the third and fourth examples above). For now, all practice should be in the context of groups of four. The four times table will be applied to four equal groups in step *1:11*.

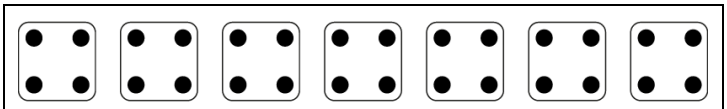
Completing multiplication equations:

'For each picture, complete the equations to show how many dots there are altogether.'

$5 \times 4 =$ ☐ $4 \times 5 =$ ☐



☐ $\times 4 =$ ☐ $4 \times$ ☐ $=$ ☐



☐ $\times$ ☐ $=$ ☐ ☐ $\times$ ☐ $=$ ☐

Representing multiplication facts:

'Eloise wrote this in her book.'

This shows 4 × 3 = 12

'Draw a picture like this to show:'

$4 \times 6 = 24$

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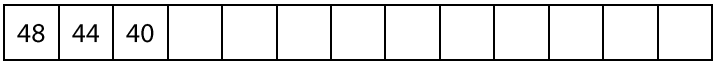
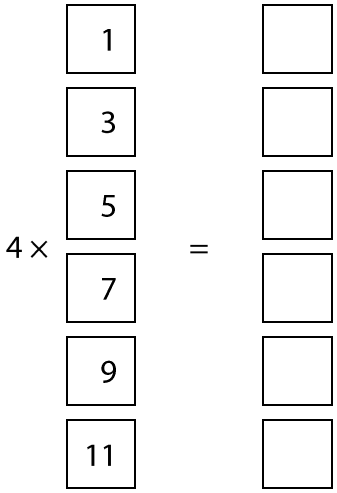
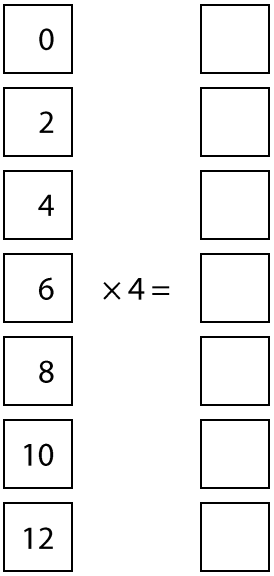
	At this stage, children can recite the four times table up to the number they need to find the answers, or use the multiplication chart for reference. Plenty of practice will be needed over an extended period until children are fluent in the isolated multiplication facts (for example, just knowing that seven fours are twenty-eight, rather than having to recite the times table up to seven fours).	Missing-number sequences/problems: *'Fill in the missing numbers.'*    

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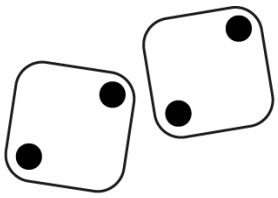
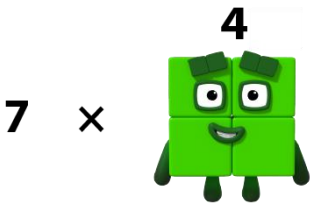
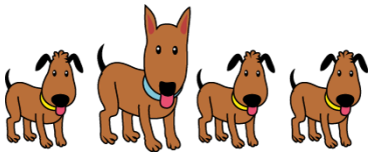
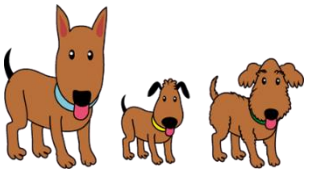
Dòng nǎo jīn:

'Which of these pictures could represent groups of four? Write two multiplication equations for each picture that represents groups of four.'

	Does (✓) or doesn't (×) represent groups of four
	
 Images © 2017 Alphablocks Ltd All Rights Reserved.	
	
	
	

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1:9

Now ask children what patterns they can see in the four times table, prompting for the following:

- The products are all even numbers.
- Products in the four times table are also products in the two times table (this will be covered in more detail in *Teaching point 2*).
- Working down the list, the product increases by four each time.

Focus in on the fact that adjacent multiples of four have a difference of four, and that this knowledge can be used to find the next or previous multiple of four from a given multiple. Use the same representations as in earlier segments to illustrate this (ratio chart, mixed-operation equations, number line and arrays).

Then challenge children to build the four times table from facts that they already know, using the rule about adjacent multiples. Before beginning, discuss why we already know the given facts/where we know them from, through applying the commutative law:

- 0×4 is known since we know that when zero is a factor, the product is zero.
- 1×4 is known since we know that when one is a factor, the product is equal to the other factor.
- 2×4, 5×4 and 10×4 are known from the two, four and ten times tables respectively.

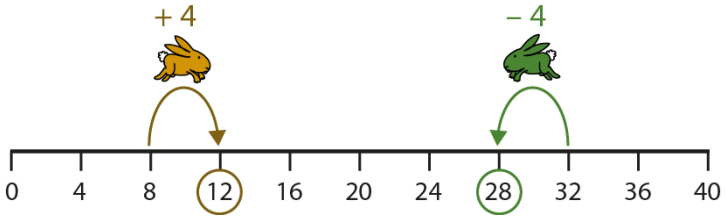
For each missing fact, encourage children to write a mixed-operation equation relating it to the next/previous fact.

Finding adjacent multiples – ratio chart and number line:

	× 4
0	0
1	4
2	8
3	
4	16
5	20
6	24
7	
8	32
9	36
10	40
11	44
12	48

↓ + 4 $\quad 3 \times 4 = 2 \times 4 + 4$

↑ − 4 $\quad 7 \times 4 = 8 \times 4 - 4$



Finding adjacent multiples – array chart:

×	1	2	3	4
1	●	●	●	●
2	●	●	●	●
3	●	●	●	●
4	●	●	●	●
5	●	●	●	●
6	●	●	●	●

$6 \times 4 = 5 \times 4 + 4$

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		Building the four times table from known facts: *'Build up the four times table from facts we already know.'* $0 \times 4 = 0$ $1 \times 4 = 4$ $2 \times 4 = 8$ $3 \times 4 =$ ☐ $4 \times 4 =$ ☐ $5 \times 4 = 20$ $6 \times 4 =$ ☐ $7 \times 4 =$ ☐ $8 \times 4 =$ ☐ $9 \times 4 =$ ☐ $10 \times 4 = 40$ $11 \times 4 =$ ☐ $12 \times 4 =$ ☐
1:10	Provide children with varied practice, based on their knowledge that adjacent multiples of four have a difference of four, including: • mixed-operation missing-number/symbol problems • contextual problems, for example: • *'In rugby, a team scores four points for winning a game. Maja's team has won their first five games. How many points will her team have altogether if they win another game?'*	Missing-number/symbol problems: *'Fill in the missing numbers.'* $3 \times 4 = 2 \times 4 +$ ☐ $6 \times 4 =$ ☐ $\times 4 + 4$ $3 \times 4 -$ ☐ $= 2 \times 4$ $6 \times 4 - 4 =$ ☐ $\times 4$ *'Fill in the missing symbols (<, > or =).'* 9×4 ◯ 8×4 9×4 ◯ $8 \times 4 + 4$

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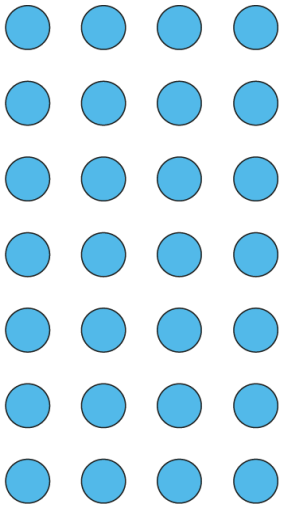
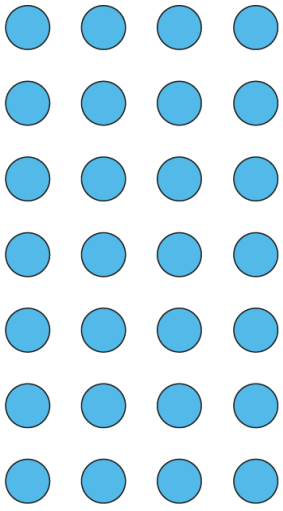
	• *'The school cook bought seven four-litre bottles of oil, then used one bottle. What volume of oil does she have now?'*	$9 \times 4 \bigcirc 9 \times 4 + 4$ $9 \times 4 \bigcirc 10 \times 4 - 4$ Dòng nǎo jīn: *'Fill in the missing numbers.'* $20 \times 4 = 80$ so $21 \times 4 = \square$ $4 \times 18 = 72$ so $4 \times 19 = \square$ $15 \times 4 = 60$ so $14 \times 4 = \square$ $4 \times 17 = 68$ so $4 \times 16 = \square$
1:11	To complete this teaching point, provide general practice that includes extending four times table facts to: • multiplication problems about four equal groups (as opposed to those about groups of four), as shown opposite and here: *'There are seven days in a week. How many days are there in four weeks?'*	Multiplication problems about four equal groups: *'Circle the groups of four and complete the sentence.'*  There are ___ groups of four. *'Circle the four equal groups and complete the sentence.'*  There are four groups of ___. $4 \times \square = 28$ $\square \times 4 = 28$

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	• non-contextual division problems (writing equations; linking multiplication and division equations; missing-number problems); use intelligent practice as shown opposite
	• contextual division problems, for example:
	  • *'Mrs Brown's class read twenty-four pages of a book. Each child read four pages. How many children read?'* (quotitive division)
	  • *'There are twenty balloons. If they are shared equally between four children, how many balloons does each child get?'* (partitive division)
	• multi-step contextual problems, for example:
	  • *'Zara is playing a dice game. She rolls three fours and a two. How much does she roll altogether?'*
	  • *'Anoki has seven bags of apples. Each bag contains four apples. If Anoki eats two of the apples, how many does he have left?'*
	  • *'Hari had twenty-four stickers shared equally between four pages of his sticker book. Then he added another sticker to each page. How many stickers does Hari have on each page?'*
	  • *'There are thirty-four children waiting to get on a fairground ride. Four children can fit in each carriage. If two children leave the queue for the ride, how many carriages will be needed?'*
		Non-contextual division problems:
		• *'Fill in the missing numbers.'*
		$3 \times 4 = \square$ $\quad 5 \times 4 = \square$
		$4 \times 3 = \square$ $\quad \square \times 4 = 20$
		$12 \div 4 = \square$ $\quad 20 \div 4 = \square$
		• *'What multiplication fact can be used to solve this division calculation?'*
		$24 \div 4 = ?$
		I can use this multiplication fact: ___ × ___ = ____

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Teaching point 2:

Products in the four times table are double the products in the two times table; products in the two times table are half of the products in the four times table.

Steps in learning

2:1

In this teaching point, the relationship between the two and four times tables is explored. Children have already made a similar comparison of the five and ten times tables (segments *2.4 Times tables: groups of 10 and of 5, and factors of 0 and 1* and *2.5 Commutativity (part 2), doubling and halving*).

Some children may already have mentioned some of the links between the two and four times tables during work on *Teaching point 1*; now, they should be given the opportunity to discuss these similarities and differences together, beginning with 'double skip counting'.

First practise counting forwards from zero in multiplies of two, and then in multiples of four. Use representations such as:

- a number line with both multiples of two and multiples of four labelled
- the Gattegno chart.

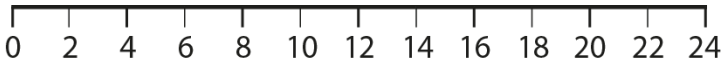
Then split the class in half, with one half counting in multiples of two and the other half counting in multiples of four, up to 24. The group counting in twos should count on every 'beat', while the group counting in fours should count on every other 'beat', such that both groups will say the multiples of four at the same time.

Then ask children what they notice, prompting for the following:

- All of the numbers said by the 'fours group' are also said by the 'twos group'.
- Not all of the numbers said by the 'twos group' are also said by the 'fours group'.
- For every number said by the 'fours group', the 'twos group' says two numbers.

After discussion, double skip count again, recording the pattern in a table as shown below.

Number line:



Gattegno chart:

1000	2000	3000	4000	5000	6000	7000	8000	9000
100	200	300	400	500	600	700	800	900
10	20	30	40	50	60	70	80	90
1	2	3	4	5	6	7	8	9

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Comparing counting in multiples of two and four:

Number	**0**	1	**2**	3	**4**	5	**6**	7	**8**	9	**10**	11	**12**	13	**14**	15	**16**	17	**18**	19	**20**	21	**22**	23	**24**
Counting in 2s	✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓
Counting in 4s	✓				✓				✓				✓				✓				✓				✓

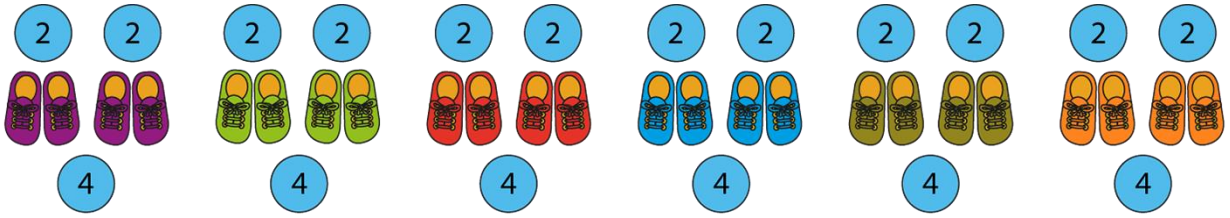
2:2

Now discuss the relationship between the number of twos and the number of fours, in terms of *groups* of two or *groups* of four, using a familiar pictorial representation. Show two pairs of shoes, and ask:

- *'How many groups of two are there?'*
- *'How many groups of four are there?'*

Use counters to represent one group of four and two groups of two. Then add another two pairs of shoes, ask the questions again and add the next set of counters. Continue until there are about six groups of two pairs, as shown below, and use a bar model to summarise the relationships.

Work towards the generalisation: ***'For every one group of four, there are two groups of two.'***



2	2	2	2	2	2	2	2	2	2	2	2
4		4		4		4		4		4	

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2:3

Now show the two and four times tables, side-by-side and ask children:

- *'What's the same?'*
- *'What's different?'*

Encourage children to use the language of factors and product, and of doubling and halving.

Prompt children to notice that the first seven products in the four times table are also found in the two times table, and ask them whether this would continue to be the case if the two times table was continued beyond 2×12. Use the generalised sentence: ***'Products in the four times table are also in the two times table.'***

Then complete missing-number problems as shown opposite, as a class, to work towards the generalisation: ***'The product of an even number and two is a product in the four times table.'***

Comparing the two and four times tables:

$2 \times 0 = 0$	$4 \times 0 = 0$
$2 \times 1 = 2$	$4 \times 1 = 4$
$2 \times 2 = 4$	$4 \times 2 = 8$
$2 \times 3 = 6$	$4 \times 3 = 12$
$2 \times 4 = 8$	$4 \times 4 = 16$
$2 \times 5 = 10$	$4 \times 5 = 20$
$2 \times 6 = 12$	$4 \times 6 = 24$
$2 \times 7 = 14$	$4 \times 7 = 28$
$2 \times 8 = 16$	$4 \times 8 = 32$
$2 \times 9 = 18$	$4 \times 9 = 36$
$2 \times 10 = 20$	$4 \times 10 = 40$
$2 \times 11 = 22$	$4 \times 11 = 44$
$2 \times 12 = 24$	$4 \times 12 = 48$

Missing-number problems:

'Fill in the missing numbers.'

0		0		0
2		4		1
4				2
6	$\times 2 =$		$= 4 \times$	3
8				4
10				5
12				6

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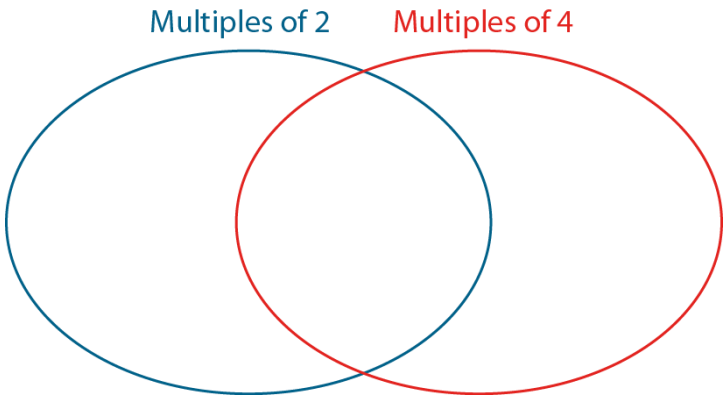
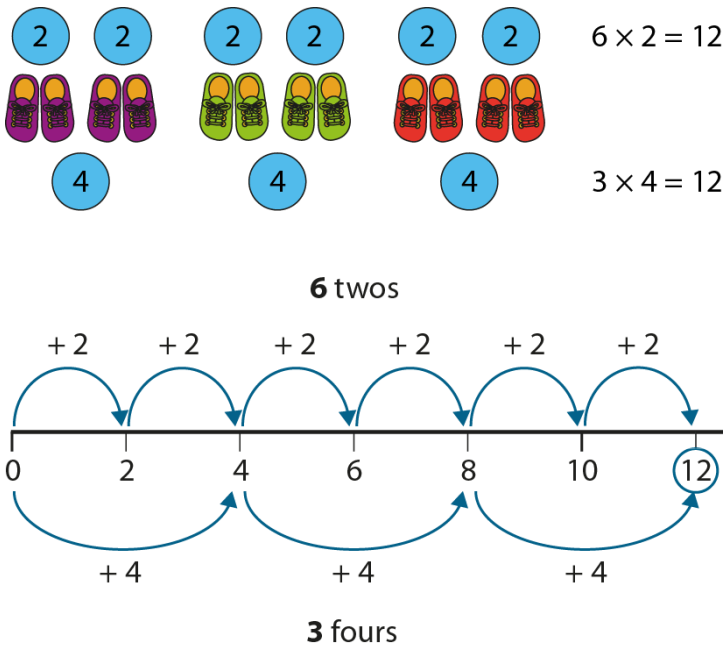
	Sort some numbers into a Venn diagram, as shown opposite. Once the numbers are sorted, ask questions to draw children's attention to the patterns and connections: • *'Which section does not have any numbers in it? Why?'* • *'What do you notice about the numbers in the section where the two sets overlap?'* • *'What do you notice about the numbers that don't go inside the circles?'* Dòng nǎo jīn: *'Rishi says all multiples of four are multiples of two.'* *'Emily says all multiples of two are multiples of four.'* *'Are they right? Why/why not?'*	Sorting numbers: *'Place these numbers in the diagram. In the overlapping section, you should place the numbers that are <u>both</u> multiples of two <u>and</u> four. Numbers that are <u>neither</u> multiples of two <u>nor</u> four should go outside the circles.'* 0 2 4 23 14 31 20 16 40 
2:4	Now use the language of doubling and halving to compare two equations with the same product (e.g. $6 \times 2 = 12$ and $3 \times 4 = 12$). You can use the shoes example from step *2:2*, along with a number line. For a given number of pairs of shoes (e.g. six): • ask how many groups of two there are (six) • represent the groups of two with two-value counters • represent the jumps of two on a number line • write a multiplication equation ($6 \times 2 = 12$). Then: • ask how many groups of four there are (three) • represent the groups of four with four-value counters • represent the jumps of four on the same number line as the twos	

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<table>
<tr>
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write a multiplication equation ($3 \times 4 = 12$).

Then compare the equations, asking children what they notice about the factors and the products. Encourage children to use the language modelled opposite to relate the two multiplication facts, and use a bar model to summarise the relationships.
Repeat for some other even quantities of pairs of shoes, until the pattern is clear.
</td>
<td>

'Three times four is equal to twelve, so double-three times two is equal to twelve.'
'Six times two is equal to twelve, so half-of-six times four is equal to twelve. '

<table>
<tr><td colspan="6">12</td></tr>
<tr><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr>
<tr><td colspan="2">4</td><td colspan="2">4</td><td colspan="2">4</td></tr>
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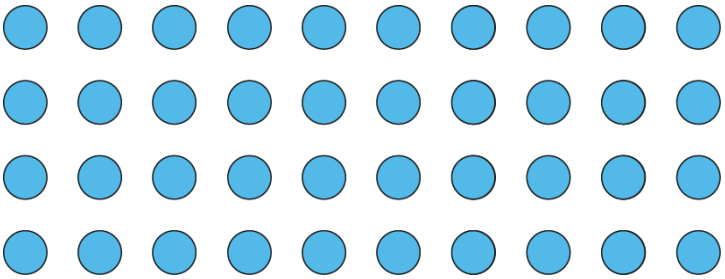
<tr>
<td>2:5</td>
<td>Provide children with some intelligent practice and contextual problems related to pairs of two and four times-table facts that have the same product. Include related pairs of division facts, supporting children to think about dividing into equal groups of two or four, or between two or four. You can also provide questions about arrays, such as that shown opposite, to support children's understanding.
Throughout, encourage children to use a known/given multiplication (or division) fact and doubling/halving strategies to find the answers, using the language in step 2:4 to explain their reasoning. (For more on doubling/halving strategies, see segment 2.5 Commutativity (part 2), doubling and halving.)</td>
<td>Array problem:


'How many groups of four are there?'

☐ × 4 = ☐

'How many groups of two are there?'

☐ × 2 = ☐</td>
</tr>
</table>

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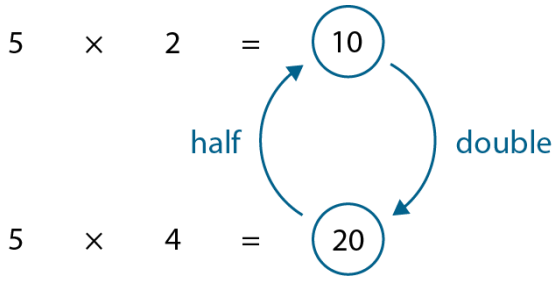
	Example word problems:
	• *'Rishi rolls <u>two</u> dice and gets two sixes. Emily rolls <u>four</u> dice and gets all threes. Emily says she got a higher total than Rishi. Is she right? Why/why not?'*
	• *'There are twenty-four marbles.'*
	  • *'If the marbles are shared equally between <u>two</u> children, how many marbles does each child get?'*
	  • *'If the marbles are shared equally between <u>four</u> children, how many marbles does each child get?'*
	(partitive division)
	• *'Twenty cupcakes are made in a bakery.'*
	  • *'How many packets of <u>two</u> can be made?'*
	  • *'How many packets of <u>four</u> can be made?'*
	(quotitive division)
	• *'Dana has a twelve-centimetre length of ribbon. She could make six <u>two</u>-centimetre lengths from this. How many <u>four</u>- centimetre lengths could she make instead?'*
	(quotitive division)
	• Dòng nǎo jīn: *'There are some cows in a field. If I can see twenty legs, how many eyes can I see?'*
		Missing-number problems:
		'Fill in the missing numbers.'
		$2 \times 2 = 1 \times 4$
		$4 \times 2 = 2 \times 4$
		$6 \times 2 = \square \times 4$
		$\square \times 2 = 4 \times 4$
		$\square \times 2 = 400 \times 4$
		$20 \times 2 = \square \times 4$
		$40 \times 2 = \square \times 4$
		$60 \times 2 = \square \times 4$
		$\square \times 2 = 40 \times 4$
		$\square \times 2 = 90 \times 4$
		$12 \div 2 = 6$
		$16 \div 2 = \square$
		$20 \div 2 = \square$
		$12 \div 4 = \square$
		$16 \div 4 = \square$
		$20 \div 4 = \square$
		True/false questions:
		'True or false?'
		$3 \times 2 + 2 = 4 \times 2$
		$3 \times 4 + 4 = 8 \times 2$

2:6	Now, shift the focus onto pairs of facts where one factor is the same and the other is either two or four, e.g.: $5 \times 2 = 10$ $5 \times 4 = 20$ Ask children what they notice, prompting them to describe how: • four is double two **and** • five-times-four is double five-times-two; • two is half of four **and** • five-times-two is half of five-times-four.	

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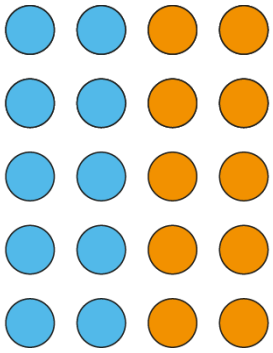
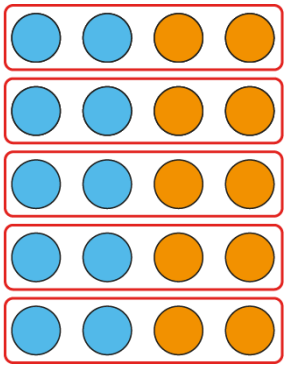
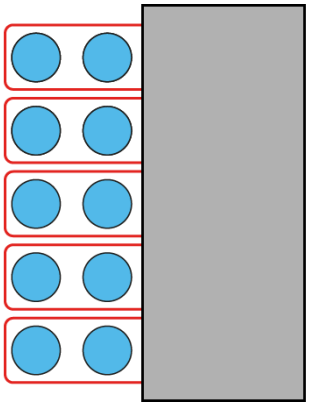
2:7	Use arrays with two different counter colours to examine *why* this is true, working through some multiplication facts in sequence. Use the following stem sentences to describe the relationships: • ***'Four is double two, so ___ fours is double ___ twos.'*** • ***'Two is half of four, so ___ twos is half of ___ fours.'*** First work through the example from the previous step (5×2 and 5×4), as shown opposite. Then work through subsequent multiplication facts (6×2 and 6×4, 7×2 and 7×4…) until children are confident with the patterns and language. (For more guidance on using the representations opposite, see segment *2.5 Commutativity (part 2), doubling and halving*, step *4:3*.)	 • *'How many fours are there?'*  *'There are five groups of four.'* $5 \times 4 = 20$ *'Five fours are twenty.'* $4 \times 5 = 20$ *'Four, five times is twenty.'* • *'How many twos are there?'*  *'There are five groups of two.'* $5 \times 2 = 10$ *'Five twos are ten.'* $2 \times 5 = 10$ *'Two, five times is ten.'* • *'Four is <u>double</u> two, so five fours is <u>double</u> five-twos.'* • *'Two is <u>half of</u> four, so five twos is <u>half of</u> five-fours.'*

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2:8	Provide children with some intelligent practice and contextual problems related to pairs of facts where one factor is the same and the other is either two or four. Include related pairs of division facts, supporting children to think about dividing into equal groups of two or four, or sharing between two or four. Example word problems: • *'Jeremy runs two kilometres each day for seven days; he runs fourteen kilometres in total. Clare runs four kilometres each of the seven days. How far does Clare run altogether?'* • *'A frog jumps two metres with every jump. A toad jumps four metres with every jump. If they've made the same number of jumps, and the toad has jumped forty metres, how far has the frog jumped?'*	Missing-number problems. *'Fill in the missing numbers.'* $2 \times 2 = \square$ $2 \times 5 = \square$ $2 \times 6 = \square$ $2 \times 4 = \square$ $4 \times 5 = \square$ $6 \times 4 = \square$ $20 \div 2 = 10$ $40 \div 4 = 10$ $16 \div 2 = \square$ $32 \div 4 = \square$ $12 \div 2 = \square$ $24 \div 4 = \square$ Dòng nǎo jīn: ♦ × ★ = 8 ♦ × ♦ = ★ ♦ = $\square$ ★ = $\square$

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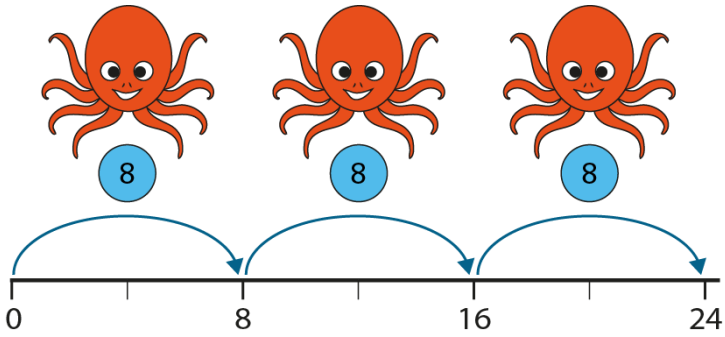
Teaching point 3:

Counting in multiples of eight can be represented by the eight times table. Adjacent multiples of eight have a difference of eight. Facts from the eight times table can be used to solve multiplication and divison problems with diffierent structures.

Steps in learning

	Guidance	Representations
3:1	The same teaching sequence that was used for the four times table (*Teaching point 1*, steps *1:2–1:11*) can be repeated for the eight times table. Similarly, the next teaching point follows the same sequence as *Teaching point 2*, but looks at the relationship between the four and eight times tables (instead of the two and four times tables). It is recommended that you ensure that children are fluent with the four times table, and are confident with links between the two and four times tables, before beginning work on the eight times table. Begin by looking at some groups of eight, linking enumerating objects in groups of eight with counting in eights, and writing the associated multiplication equations (write two equations for each example, as shown opposite). Use contexts that children already associate with eight; for example, legs on a spider or tentacles on an octopus. Use eight-value counters alongside each context to support the idea of unitising in eight, and use a number line with the multiples of eight highlighted for skip-counting support. For each example, ask children to describe what each number in the equation represents, and to use the language of factor and product to describe the equations (see step *1:2* for more guidance).	*'How many tentacles are there? Count in groups of eight.'*  • *'Eight, sixteen, twenty-four. There are twenty-four tentacles.'* • *'There are three groups of eight; there are twenty-four altogether.'* • *'There are eight, three times; there are twenty-four altogether.'* $3 \times 8 = 24$ $8 \times 3 = 24$ • *'Three is a factor.'* • *'Eight is a factor.'* • *'The product of three and eight is twenty-four.'* • *'Twenty-four is the product of three and eight.'*

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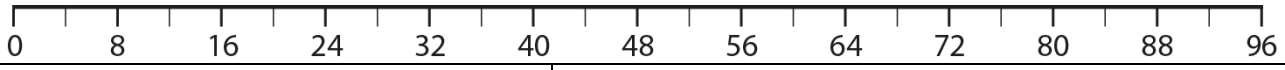
3:2 Include writing the equations $0 \times 8 = 0$ and $8 \times 0 = 0$. By this point, children will know that when one of the factors is zero, the product will be zero. Refer to step *1:3* for more guidance if necessary.

3:3 Practise skip counting, forwards and backwards in eights between 0 and 96, regularly outside the main maths lesson, so that children begin to develop fluency with this counting sequence before moving onto the next step. Use familiar representations such as a number line and the Gattegno chart.

Number line:

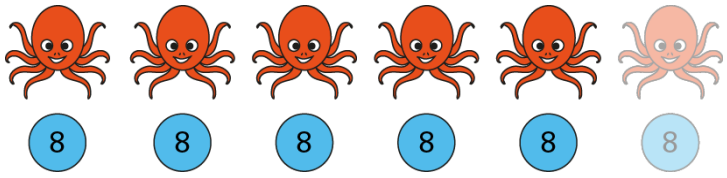


3:4 Now, using a familiar context, work systematically to construct the eight times table, beginning with zero eights.

Use a ratio chart to record the number of groups and the product as you go, and also write the multiplication equations (two equations for each times-table fact). Use the same form of language as described in step *1:4*, for example:

- $5 \times 8 = 40$
  - *'Five groups of eight is equal to forty.'*
  - *'Five times eight is equal to forty.'*
- $8 \times 5 = 40$
  - *'Eight, five times is equal to forty.'*
  - *'Eight times five is equal to forty.'*

Building up the eight times table:



$0 \times 8 = 0$	$8 \times 0 = 0$
$1 \times 8 = 8$	$8 \times 1 = 8$
$2 \times 8 = 16$	$8 \times 2 = 16$
$3 \times 8 = 24$	$8 \times 3 = 24$
$4 \times 8 = 32$	$8 \times 4 = 32$
$5 \times 8 = 40$	$8 \times 5 = 40$
$6 \times 8 = 48$	**$8 \times 6 = 48$**

Number of octopuses	Total number of tentacles
0	0
1	8
2	16
3	24
4	32
5	40
6	**48**

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3:5

Once the ratio chart and full set of equations are complete, ask children questions, encouraging them to use the chart/equations for support, for example:

- *'If there are nine octopuses, how many tentacles are there altogether?'*
- *'How many octopuses are there if there are forty-eight tentacles?'*
- *'If the product is eighty, what are the factors?'*
- *'Why are eight times four and four times eight both equal to thirty-two?'*
- Dòng nǎo jīn: *'Jon says that he can subtract two from all the products in the ten times table to create the eight times table, because there are two less in each group. Is he right?'*

Complete ratio chart and eight times table:

Number of octopuses	Total number of tentacles
0	0
1	8
2	16
3	24
4	32
5	40
6	48
7	56
8	64
9	72
10	80
11	88
12	96

$0 \times 8 = 0$	$8 \times 0 = 0$
$1 \times 8 = 8$	$8 \times 1 = 8$
$2 \times 8 = 16$	$8 \times 2 = 16$
$3 \times 8 = 24$	$8 \times 3 = 24$
$4 \times 8 = 32$	$8 \times 4 = 32$
$5 \times 8 = 40$	$8 \times 5 = 40$
$6 \times 8 = 48$	$8 \times 6 = 48$
$7 \times 8 = 56$	$8 \times 7 = 56$
$8 \times 8 = 64$	$8 \times 8 = 64$
$9 \times 8 = 72$	$8 \times 9 = 72$
$10 \times 8 = 80$	$8 \times 10 = 80$
$11 \times 8 = 88$	$8 \times 11 = 88$
$12 \times 8 = 96$	$8 \times 12 = 96$

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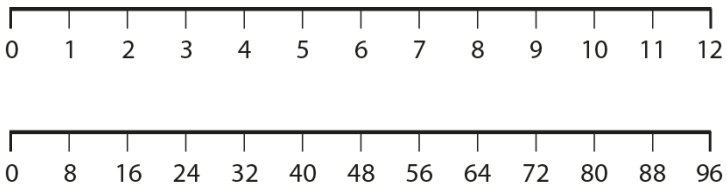
3:6	Now practise chanting the eight times table, with the written times table for support, using a variety of representations, including: • stacked number lines (as shown opposite) • the Gattegno chart • concrete representations • pictorial representations. Use the following language: • *'One group of eight is equal to eight.' 'Two groups of eight is equal to sixteen…'* • *'One times eight is equal to eight.' 'Two times eight is equal to sixteen…'* then shortening to • *'One eight is eight, two eights are sixteen…'* and • *'Eight, one time is equal to eight...' 'Eight, two times is equal to sixteen...'* • *'Eight times one is equal to eight...' 'Eight times two is equal to sixteen...'* Regular practice should be undertaken, including outside the main maths lesson, until children are fluent.	

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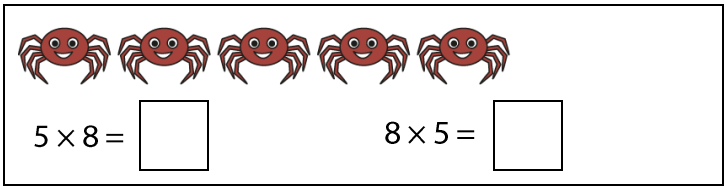
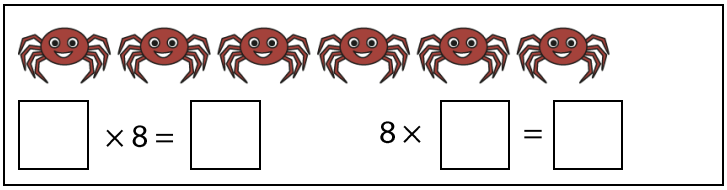
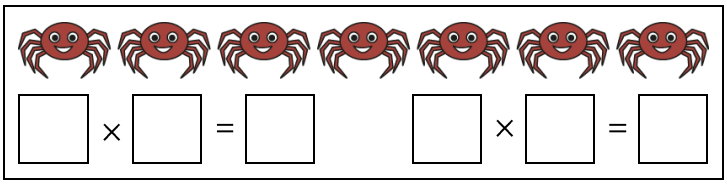
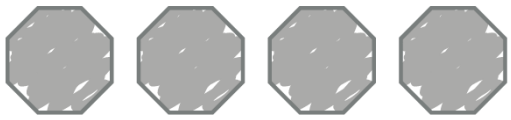
3:7	Provide practice, similar to that in step *1:8*. For now, all practice should be in the context of groups of eight. The eight times table will be applied to eight equal groups in step *3:10*. Example word problems: • *'What is the product of "4" and "8"?'* • *'Deepti buys nine books. Each book costs £8. How much does she spend altogether?'* • *'There are eleven children. Each child has eight counters. How many counters is this altogether?'* Children should write a multiplication equation for each problem, rather than simply writing the product.	Completing multiplication equations: *'For each picture, complete the equations to show how many legs there are altogether.'*  $5 \times 8 =$ ☐ $8 \times 5 =$ ☐  ☐ $\times 8 =$ ☐ $8 \times$ ☐ $=$ ☐  ☐ $\times$ ☐ $=$ ☐ ☐ $\times$ ☐ $=$ ☐ Representing multiplication facts: *'Hari wrote this in his book.'*  This shows 4 × 8 = 32 *'Draw a picture like this to show:'* $7 \times 8 = 56$

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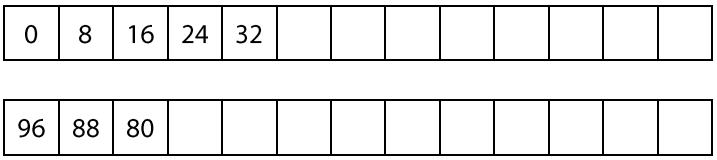
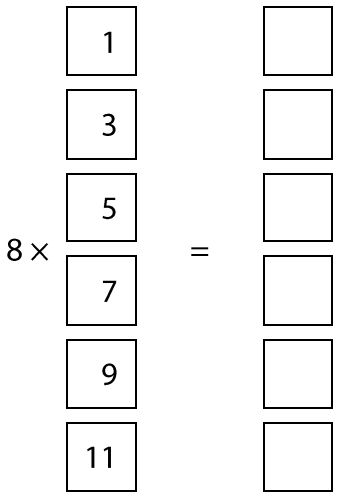
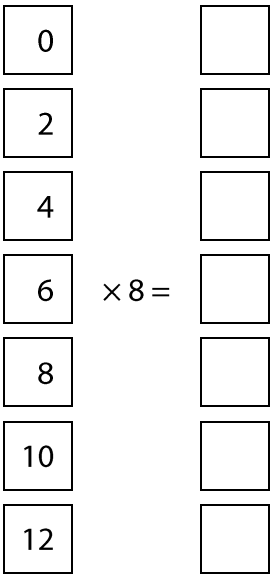
	At this stage, children can recite the eight times table up to the number they need to find the answers, or use the multiplication chart for reference. Plenty of practice will be needed over an extended period until children are fluent in the isolated multiplication facts (for example, just knowing that seven times eight is fifty-six, rather than having to recite the times table up to seven eights).	Missing-number sequences/problems: *'Fill in the missing numbers.'*   

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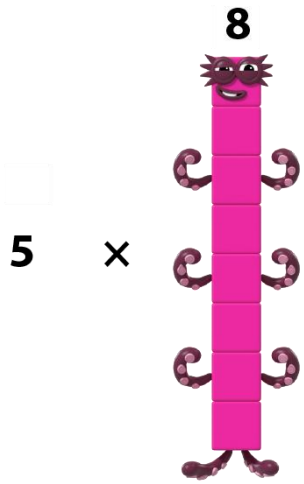
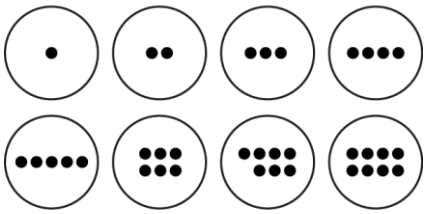
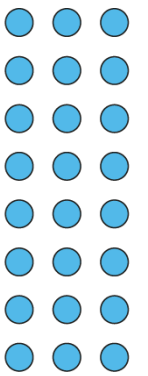
Dòng nǎo jīn:

'Which of these pictures could represent groups of eight? Write two multiplication equations for each picture that represents groups of eight.'

	Does (✓) or doesn't (×) represent groups of 8
	
 Images © 2017 Alphablocks Ltd All Rights Reserved.	
	
	
	

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3:8

Similarly to step *1:9*, ask children what patterns they can see in the eight times table, prompting for the following:

- The products are all even numbers.
- Products in the eight times table are also products in the two times table and the four times table (this will be covered in more detail in *Teaching points 4* and *5*).
- Working down the list, the product increases by eight each time.

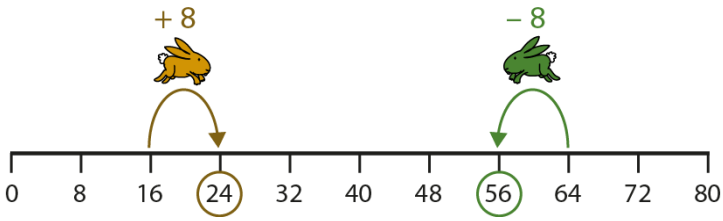
Focus in on the fact that adjacent multiples of eight have a difference of eight, and that this knowledge can be used to find the next or previous multiple of eight from a given multiple. Use the same representations as before to illustrate this (ratio chart, mixed-operation equations, number line and arrays).

Finding adjacent multiples – ratio chart and number line:

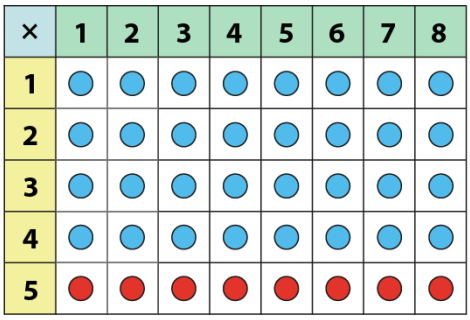
	× 8
0	0
1	8
2	16
3	
4	32
5	40
6	48
7	
8	64
9	72
10	80
11	88
12	96

↓ + 8 $3 \times 8 = 2 \times 8 + 8$

↑ − 8 $7 \times 8 = 8 \times 8 - 8$



Finding adjacent multiples – array chart:



$4 \times 8 = 5 \times 8 - 8$

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	Then challenge children to build the eight times table from facts that they already know, using the rule about adjacent multiples, and writing mixed operation equations to represent the relationships. Before beginning, discuss why we already know the given facts/where we know them from, through applying the commutative law.	Building the eight times table from known facts: *'Build up the eight times table from facts we already know.'* $0 \times 8 = 0$ $1 \times 8 = 8$ $2 \times 8 = 16$ $3 \times 8 =$ ☐ $4 \times 8 = 32$ $5 \times 8 = 40$ $6 \times 8 =$ ☐ $7 \times 8 =$ ☐ $8 \times 8 =$ ☐ $9 \times 8 =$ ☐ $10 \times 8 = 80$ $11 \times 8 =$ ☐ $12 \times 8 =$ ☐

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3:9	In the same way as in step *1:10*, provide children with varied practice based on their knowledge that adjacent multiples of eight have a difference of eight. Example word problems: • *'Nick buys seven packs of eight pencils and gets one free pack. How many pencils does he have altogether?'* • *'Jordan buys six packs of eight pencils and gives one pack to his friend. How many pencils does he have left?'*	Missing-number/symbol problems: *'Fill in the missing numbers.'* $3 \times 8 = 2 \times 8 + \square$ $\quad 6 \times 8 = \square \times 8 + 8$ $3 \times 8 - \square = 2 \times 8$ $\quad 6 \times 8 - 8 = \square \times 8$ *'Fill in the missing symbols (<, > or =).'* $4 \times 8 \bigcirc 5 \times 8 - 5$ $4 \times 8 \bigcirc 5 \times 8 - 8$ $4 \times 8 \bigcirc 3 \times 8 + 3$ $4 \times 8 \bigcirc 3 \times 8 + 8$ Dòng nǎo jīn: *'Fill in the missing numbers.'* $20 \times 8 = 160$ so $21 \times 8 = \square$ $8 \times 18 = 144$ so $8 \times 19 = \square$ $15 \times 8 = 120$ so $18 \times 8 = \square$ $8 \times 17 = 136$ so $8 \times 16 = \square$

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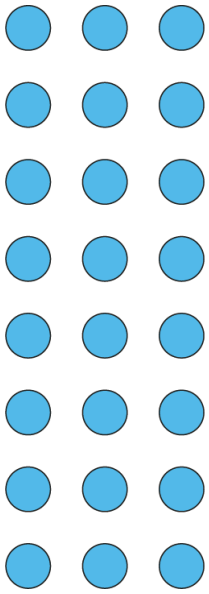
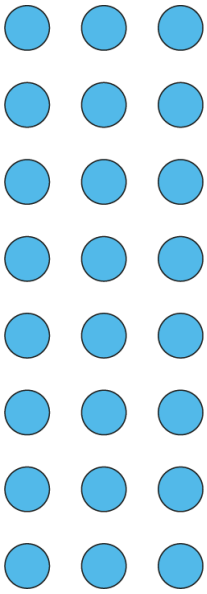
3:10

To complete this teaching point, provide general practice that includes extending eight times table facts to:

- multiplication problems about eight equal groups (as opposed to those about groups of eight), as shown opposite and here: *'Eight children each have six stickers. How many stickers do they have altogether?'*
- non-contextual division problems (writing equations; linking multiplication and division equations; missing-number problems); use intelligent practice as shown opposite
- contextual division problems, for example:
  - *'There are eight boxes of breakfast cereal in one multipack. Mr Smith needs seventy boxes of cereal for the school trip. How many multipacks should he buy?'* (quotitive division)
  - *'Forty conkers are shared equally between eight children. How many conkers does each child get?'* (partitive division)
- multi-step contextual problems, for example:
  - *'There are three packs of eight pencils. Two pencils in one of the packs are broken. How many unbroken pencils are there?'*
  - *'Thirty-two balloons were shared equally between eight children. Then another balloon was given to each child. How many balloons does each child have?'*

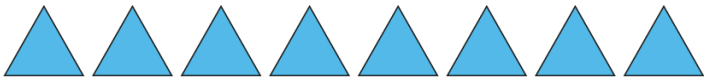
Multiplication problems about eight equal groups

'Circle the groups of eight and complete the sentence.'	*'Circle the eight equal groups and complete the sentence.'*
	
There are ___ groups of eight.	There are eight groups of ___.

$8 \times \square = 24$

$\square \times 8 = 24$

- *'Does this represent a fact in the eight times table?'*



Division problems:

- *'Fill in the missing numbers.'*

$8 \times 3 = \square$ $\quad$ $5 \times 8 = \square$

$3 \times 8 = \square$ $\quad$ $\square \times 8 = 40$

$24 \div 8 = \square$ $\quad$ $40 \div 8 = \square$

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Number of spiders	0	1		3	4		6	
Total number of legs		8	16		32	40		56

- *'What multiplication fact can be used to solve this division calculation?'*

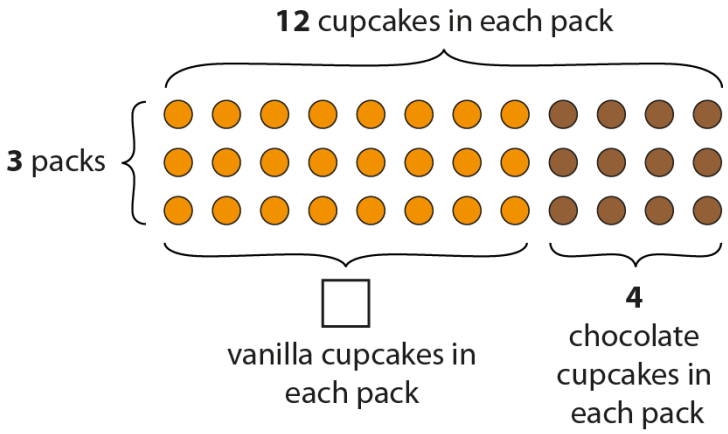
  48 ÷ 8 = ?

  I can use this multiplication fact: ___ × ___ = ____

Multi-step contextual problem:

'There are twelve cupcakes in a pack. In each pack, four cupcakes are chocolate and the rest are vanilla.'

- *How many vanilla cupcakes are there in three packs?'*



Total number of vanilla cupcakes = ☐

- *'How many vanilla cupcakes are there in six packs?'*

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Teaching point 4:

Products in the eight times table are double the products in the four times table; products in the four times table are half of the products in the eight times table. Products that are in the two, four and eight times tables share the same factors.

Steps in learning

4:1

This teaching point builds on the links that have already been made between the two and four times tables and extends these links to the four and eight times tables. Steps *4:1–4:8* follow the same progression as for *Teaching point 2*, so guidance here is kept brief; for more detail refer to *Teaching point 2*. The remaining steps of this teaching point bring together the relationships between the two, four and eight times tables.

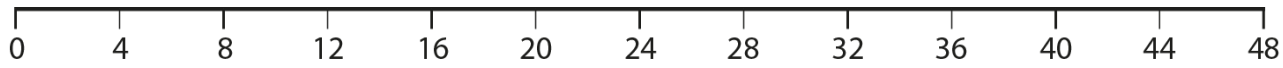
Since children have worked through a similar sequence for the five and ten times tables, and for the two and four times tables, you will probably be able to work through steps *4:1–4:8* more quickly, as children are likely to grasp the patterns more easily.

First practise counting forwards from zero in multiplies of four, then in multiples of eight. Then split the class in half and 'double skip count' in fours and eights, from 0 to 48, as described in step *2:1*. Use representations such as:

- a number line with both multiples of two and multiples of four labelled
- the Gattegno chart.

Discuss and record the pattern in a table as shown below.

Number line:



Gattegno chart:

1000	2000	3000	4000	5000	6000	7000	8000	9000
100	200	300	400	500	600	700	800	900
10	20	30	40	50	60	70	80	90
1	2	3	4	5	6	7	8	9

Comparing counting in multiples of four and eight:

Number	**0**	1	2	3	**4**	5	6	7	**8**	9	10	11	**12**	13	14	15	**16**	17	18	19	**20**	21	22	23	**24**
Counting in 4s	✓				✓				✓				✓				✓				✓				✓
Counting in 8s	✓								✓								✓								✓

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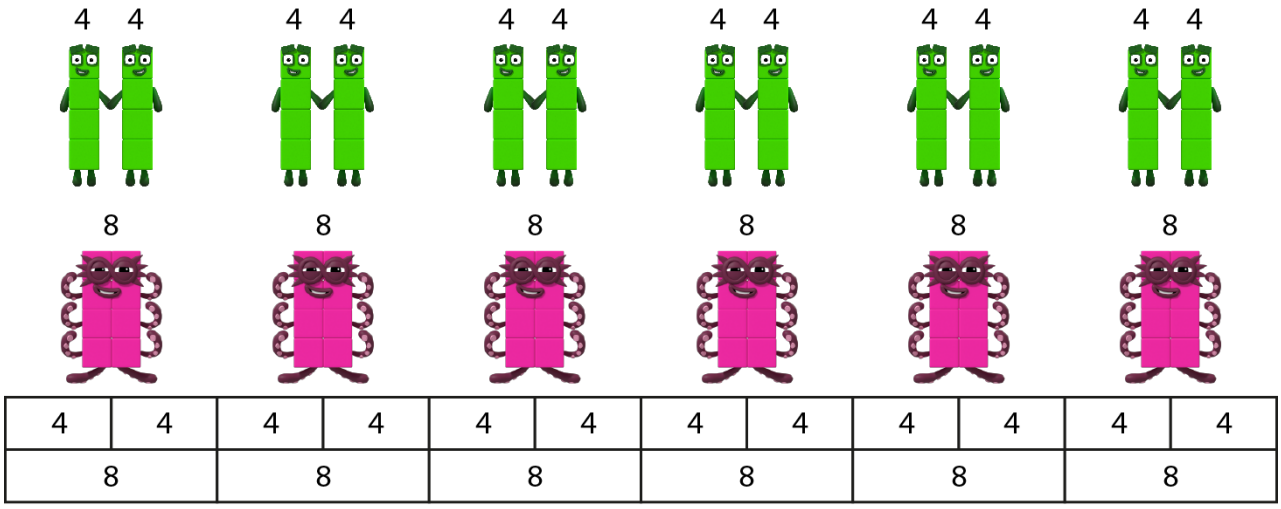
4:2

Now discuss the relationship between the number of fours and the number of eights in terms of *groups* of four or *groups* of eight, using a familiar pictorial representation. Work upwards from two fours/one eight in a similar way to that described in step *2:2*.

Work towards the generalisation: ***'For every one group of eight, there are two groups of four.'***

You can use the bar model to summarise the relationships.



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4:3

Now show the four and eight times tables and ask children:

- *'What's the same?'*
- *'What's different?'*

Encourage children to use the language of factors and product, and of doubling and halving, including the following generalisations:

- ***'Products in the eight times table are also in the four times table.'***
- ***'The product of an even number and four is a product in the eight times table.'***

Comparing the four and eight times tables:

$4 \times 0 = 0$	$8 \times 0 = 0$
$4 \times 1 = 4$	$8 \times 1 = 8$
$4 \times 2 = 8$	$8 \times 2 = 16$
$4 \times 3 = 12$	$8 \times 3 = 24$
$4 \times 4 = 16$	$8 \times 4 = 32$
$4 \times 5 = 20$	$8 \times 5 = 40$
$4 \times 6 = 24$	$8 \times 6 = 48$
$4 \times 7 = 28$	$8 \times 7 = 56$
$4 \times 8 = 32$	$8 \times 8 = 64$
$4 \times 9 = 36$	$8 \times 9 = 72$
$4 \times 10 = 40$	$8 \times 10 = 80$
$4 \times 11 = 44$	$8 \times 11 = 88$
$4 \times 12 = 48$	$8 \times 12 = 96$

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As a class, sort some numbers into a Venn diagram, as shown opposite, and discuss the patterns and connections (for more information on prompting questions to use, see step *2:3*).

Missing-number problems:

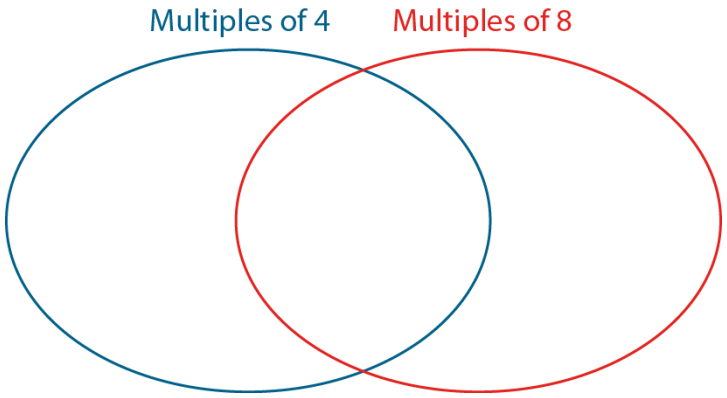
'Fill in the missing numbers.'

0		0		0
2		8		1
4				2
6	× 4 =		= 8 ×	3
8				4
10				5
12				6

Sorting numbers:

'Place these numbers in the diagram. In the overlapping section, you should place the numbers that are <u>both</u> multiples of four <u>and</u> eight. Numbers that are <u>neither</u> multiples of four <u>nor</u> eight should go outside the circles.'

0	4	8	45	12	31	24	44	40



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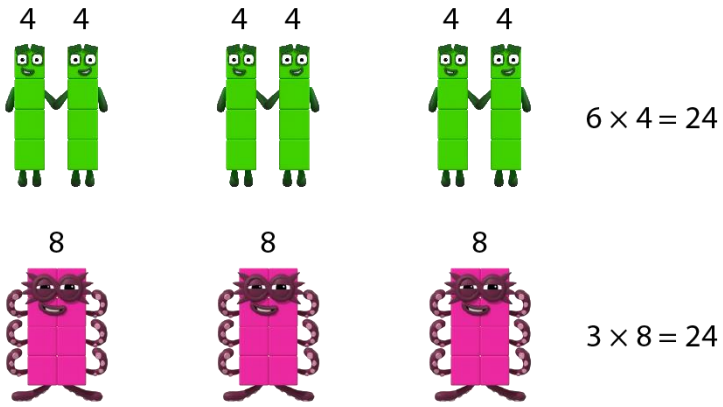
4:4

Now use the language of doubling and halving to compare two equations with the same product (e.g. $6 \times 4 = 24$ and $3 \times 8 = 24$). You can use the Numberblocks example from step *4:2*, along with a number line.

For a given number represented with Numberblock characters (for example six fours vs three eights), compare the number of groups of four and of eight, write the corresponding multiplication equations, and discuss the relationship between the equations (for more guidance, see step *2:4*).



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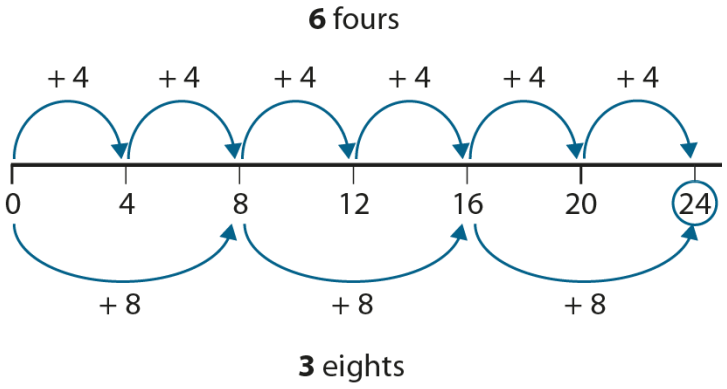

- *'Three times eight is equal to twenty-four, so double-three times four is equal to twenty-four.'*
- *'Six times four is equal to twenty-four, so half-of-six times eight is equal to twenty-four.'*

24					
4	4	4	4	4	4
8		8		8	

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4:5	Provide children with some intelligent practice and contextual problems related to pairs of two and four times-table facts that have the same product. Include related pairs of division facts. A bar model could be used to show the halving and doubling relationship, but it should not be relied upon as a method to calculate the answer. Children should draw on their known doubling/halving facts and strategies (segment *2.5 Commutativity (part 2), doubling and halving*). Example word problems: • *'There is a twenty-four metre length of bunting for the school fair. Six four-metre lengths could be made. How many eight-metre lengths could be made?'* • *'Batteries are sold in packs of four or eight. I want to buy six packs of eight batteries, but the shop has run out of eight-packs. How many packs of four batteries should I buy to get the same number of batteries altogether?'* • *'Lily is given pocket money every Saturday. Her mum gives her £4 and her dad gives her £4. How many weeks will it take her to save up for a bike that costs £84?'*	Missing-number problems: *'Fill in the missing numbers.'* $1 \times 8 = 2 \times 4$ $2 \times 8 = \square \times 4$ $3 \times 8 = \square \times 4$ $4 \times 8 = 8 \times \square$ $\square \times 8 = 10 \times 4$ $10 \times 8 = \square \times 4$ $\square \times 8 = 40 \times 4$ $30 \times \square = 60 \times 4$ $40 \times 8 = \square \times 4$ $\square \times 8 = 100 \times 4$ $24 \div 4 = 6$ $32 \div 4 = \square$ $40 \div 4 = \square$ $24 \div 8 = \square$ $32 \div 8 = \square$ $40 \div 8 = \square$ $3 \times 4 + 4 = \square \times 8$ $5 \times 4 + 8 = \square \times 4$ $8 \times 4 + 4 + 4 = \square \times 4$ $8 \times 4 + 4 + 4 = \square \times 8$ True/false questions: *'True or False?'* $5 \times 4 + 4 = 6 \times 4$ $7 \times 4 + 8 = 9 \times 4$

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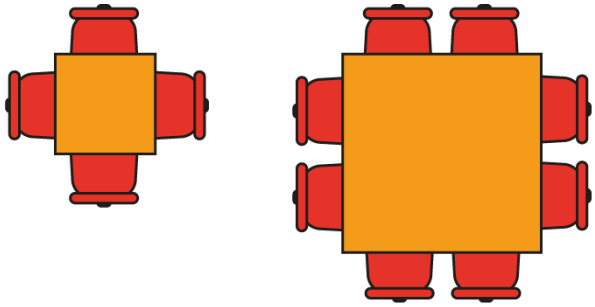
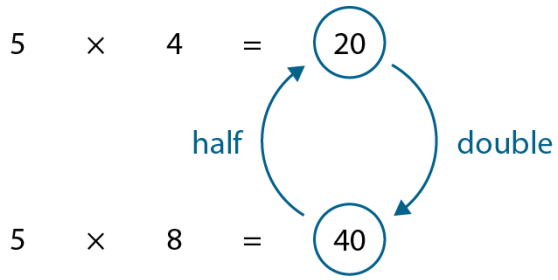
		Contextual problems:
		• *'A class of children can sit around tables in eights or in fours. If they sit round tables in fours, they need six tables. How many tables will they need if they sit in eights? '*
		 
		• *'There are thirty-two children in a class.*
		  • *How many tables will be needed if each table can seat <u>four</u> children?*
		  • *How many tables would be needed if each table can seat <u>eight</u> children?'*

4:6	In a similar way to step *2:6*, now shift the focus onto pairs of facts where one factor is the same and the other is either four or eight, e.g.: $5 \times 4 = 20$ $5 \times 8 = 40$ Ask children to describe what they notice about the equations.	

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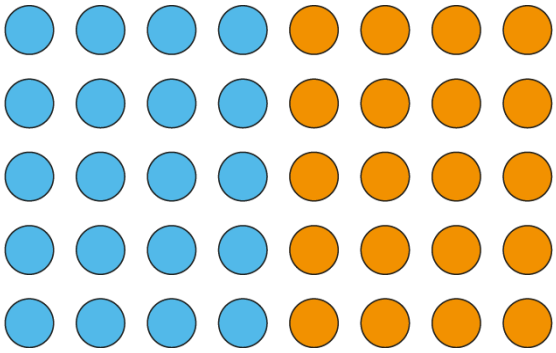
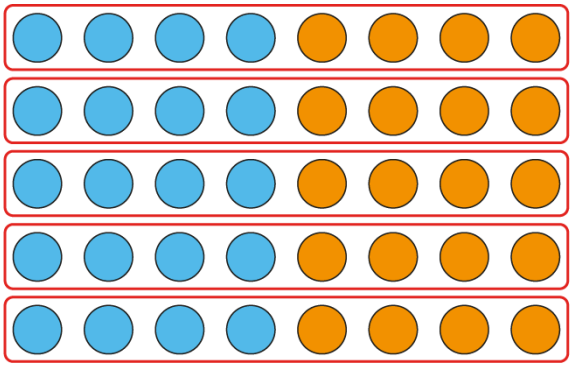
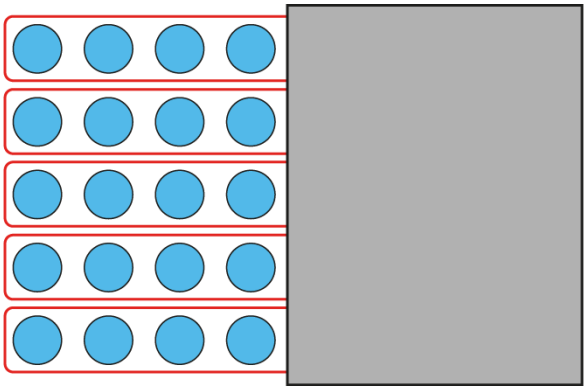
4:7	As in step *2:7*, use arrays with two different counter colours to examine *why* this is true, and use the following stem sentences to describe the relationships: • ***'Eight is double four, so ___ eights is double ___ fours.'*** • ***'Four is half of eight, so ___ fours is half of ___ eights.'***	 • *'How many eights are there?'*  *'There are five groups of eight.'* $5 \times 8 = 40$ *'Five eights are forty.'* $8 \times 5 = 40$ *'Eight, five times is forty.'* • *'How many fours are there?'*  *'There are five groups of four.'* $5 \times 4 = 20$ *'Five fours are twenty.'* $4 \times 5 = 20$ *'Four, five times is twenty.'* • *'Eight is <u>double</u> four, so five eights is <u>double</u> five-fours.'* • *'Four is <u>half of</u> eight, so five fours is <u>half of</u> five-eights.'*

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<table>
<tr>
<td>4:8</td>
<td>Provide children with some intelligent practice and contextual problems related to pairs of facts where one factor is the same and the other is either four or eight. Include related pairs of division facts.

Example word problems:

• 'Clare runs four kilometres each day for seven days; she runs twenty-eight kilometres in total. Tasha runs eight kilometres each of the seven days. How far does Tasha run altogether?'
• 'A frog jumps four metres with every jump. A toad jumps eight metres with every jump. If they've made the same number of jumps, and the toad has jumped eighty metres, how far has the frog jumped?'</td>
<td>Missing-number problems.

'Fill in the missing numbers.'

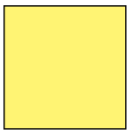
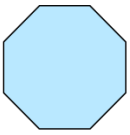
2 × 4 = ☐ 4 × 5 = ☐
2 × 8 = ☐ 8 × 5 = ☐

7 × 4 = ☐ 20 × 4 = ☐
7 × 8 = ☐ 20 × 8 = ☐

40 ÷ 4 = 10 40 ÷ 8 = 5
32 ÷ 4 = ☐ 32 ÷ 8 = ☐
24 ÷ 4 = ☐ 24 ÷ 8 = ☐

Contextual problem:

'Felicity and Hamid have some shapes.'

  square
  octagon

• 'Felicity has six squares. How many corners does she have altogether?'
• 'Hamid has the same number of octagons. How many corners does he have altogether?'</td>
</tr>
<tr>
<td>4:9</td>
<td colspan="2">The rest of this teaching point combines knowledge of patterns in the two, four and eight times tables, and focuses on the products common to these times tables. Note that the language of common factors and lowest common multiples are not used here; these ideas are considered in segment 2.21 Factors, multiples, prime numbers and composite numbers.

Compare the two, four and eight times tables using:

• 'triple skip counting'; split the class into three, with one group counting in twos, one in fours and one in eights, using a number line or Gattegno chart for support
• multiplication charts
• a number line showing multiples of two, four and eight
• a multiples grid similar to that used in steps 2:1 and 2:4.</td>
</tr>
</table>

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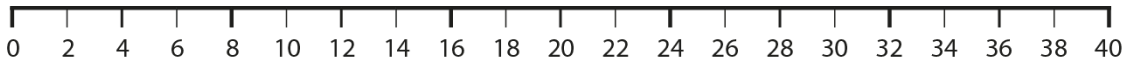
Remind children of the following two known generalisations from earlier in the segment:

- ***'Products in the four times table are also in the two times table.'***
- ***'Products in the eight times table are also in the four times table.'***

Then prompt children to notice that: ***'Products in the eight times table are also in the two times table.'***

(Children have already noted that products in the eight times table are even numbers, so this can be included as part of the reasoning).

Number line:



Multiples chart for two, four and eight:

Number	**0**	1	**2**	3	**4**	5	**6**	7	**8**	9	**10**	11	**12**	13	**14**	15	**16**	17	**18**	19	**20**	21	**22**	23	**24**
Multiples of 2	✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓
Multiples of 4	✓				✓				✓				✓				✓				✓				✓
Multiples of 8	✓								✓								✓								✓

4:10

Now focus further on numbers that are multiples of two, four and eight (same product).

Using manipulatives, such as counting sticks, take each multiple of eight in turn and ask children to explore the different ways that they can be grouped equally. When bringing together children's grouping suggestions as a class, work pictorially and circle the groups to make the number of groups and the size of the groups clear. Write multiplication equations to go with each grouping.

Draw attention to the fact that, for each multiple of eight, the manipulatives can be grouped in (ones), twos, fours and eights. Ask children to describe the relationships between the group size and the number of groups in each case, using their doubling and halving language from earlier in this teaching point. For a given number of sticks, for example 24, draw out that because eight is a factor, then two and four are also factors, because each group of eight contains groups of two or groups of four. Groups of two, four and eight can be independently combined to make 24, and conversely 24 can be divided into equal groups of two, four or eight.

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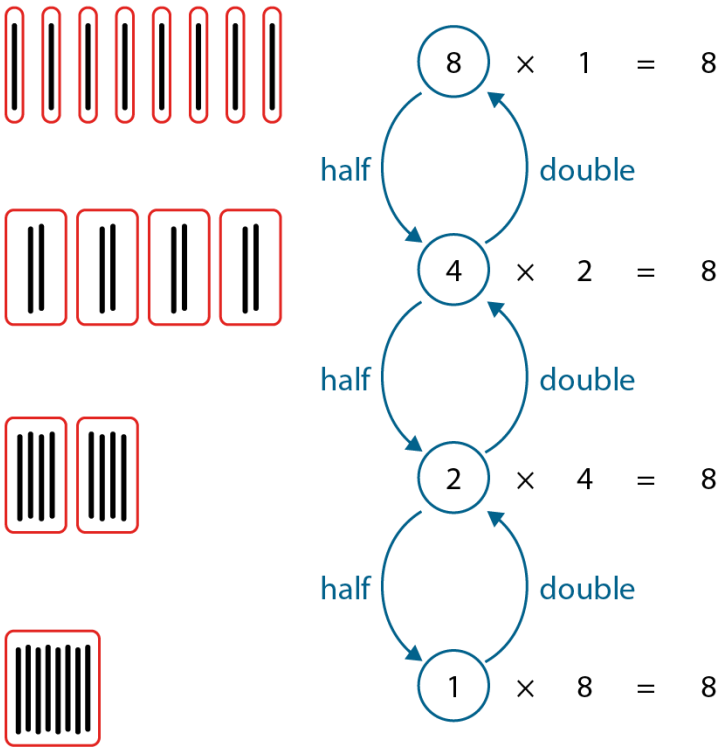
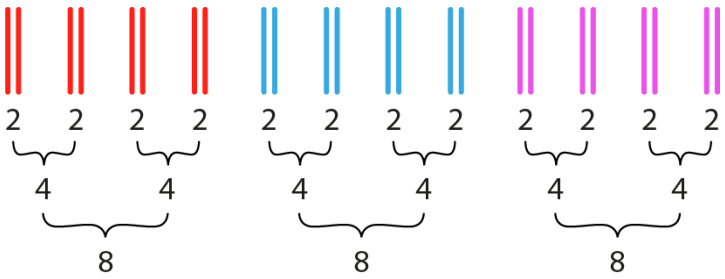
Grouping eight sticks:



Seeing the groups of two and four in groups of eight:



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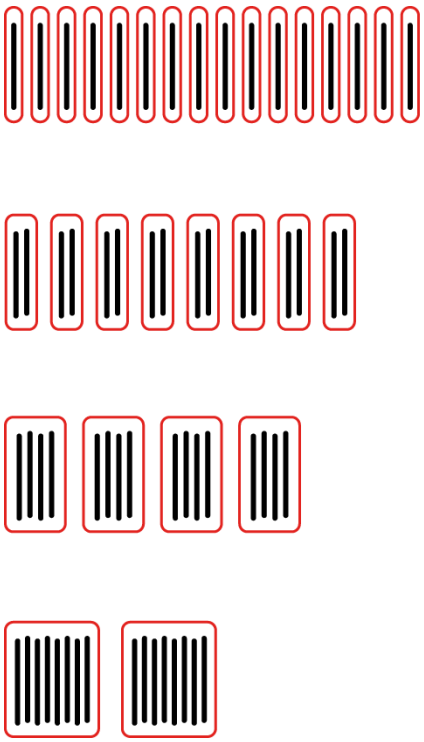
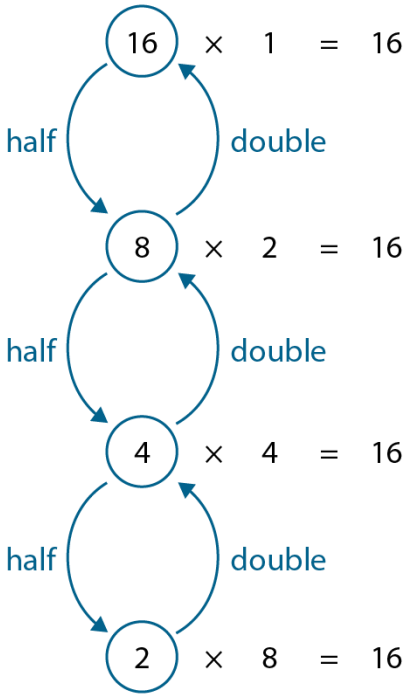
Grouping sixteen sticks:





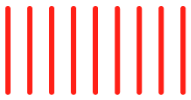
4:11

Once children are secure in their understanding that a multiple of eight is also a multiple of two and of four, explore a number that is *not* a multiple of two, four or eight. This exploration of a 'non-concept' helps to deepen children's understanding, and also provides some experience with remainders before it is explored formally in segment *2.12 Division with remainders.*

Begin with an example with a remainder of one, such as that shown opposite. Ask children:

- to make groups of two
- to describe how many groups there are and how many sticks are left over
- to write a mixed operation equation and the language of remainders to represent this (as introduced in segment *2.6 Structures: quotitive and partitive division, Teaching point 1*).

- *'Basil has nine sticks.'*



- *'How many two-stick mountain shapes can he make?'*

- *'Basil can make four two-stick mountain shapes. There is one stick left over.'*

$9 = 4 \times 2 + 1$

- *'Nine is divided into four groups of two with a remainder of one.'*
- *'The "9" represents the total number of sticks.'*
- *'The "4 × 2" represents the four shapes each made of two sticks.'*
- *The "1" represents the remaining stick.'*

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Then repeat, making groups of four, then of eight from the same set of sticks. Discuss why there is always one stick left over when nine sticks are used to make shapes from two, four or eight sticks.

Present an incorrectly solved problem such as that shown opposite (Nickolay's problem), in which the remainder is larger than group size. This allows children to discover for themselves that the remainder cannot be greater than the number of items in each group; in this situation another group of four sticks can be used to make a square. Although Nickolay's equation ($15 = 2 \times 4 + 7$) accurately represents his solution to the problem, it does not represent the correct solution in which the greatest number of squares has been made from the sticks ($15 = 3 \times 4 + 3$).

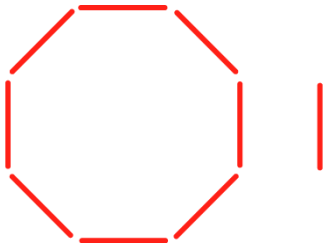
- *'How many squares can he make?'*

  - *'He can make two squares. There is still one stick left over.'*

    $9 = 2 \times 4 + 1$

  - *'Nine is divided into two groups of four with a remainder of one.'*
  - *'The "9" represents the total number of sticks.'*
  - *'The "2 × 4" represents the two shapes each made of four sticks.'*
  - *'The "1" represents the remaining stick.'*

- *'How many octagons can he make?'*

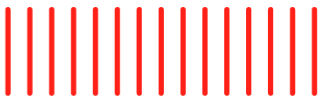


  - *'He can make one octagon. There is still one stick left over.'*

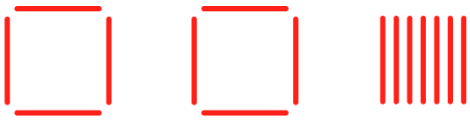
    $9 = 1 \times 8 + 1$

  - *'Nine is one group of eight with a remainder of one.'*
  - *'The "9" represents the total number of sticks.'*
  - *'The "1 × 8" represents the one shape made of eight sticks.'*
  - *'The "1" represents the remaining stick.'*

- *'Nickolay has 15 sticks.'*



'He makes as many squares as possible.'



$15 = 2 \times 4 + 7$

'Do you agree with his representation and equation?'

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4:12

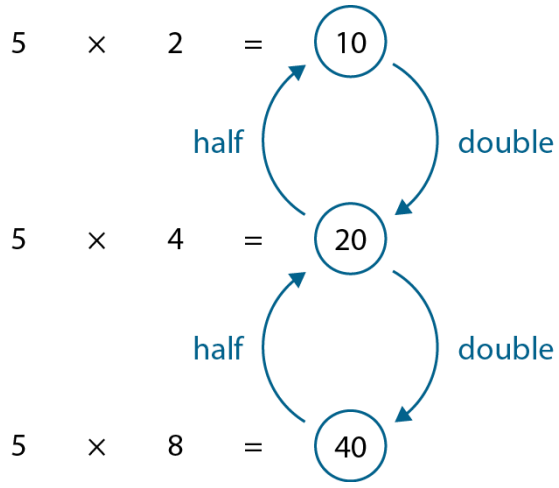
In a similar way to steps *2:6* and *4:6*, now shift the focus onto triplets of facts where one factor is the same and the other is either two, four or eight, e.g.:

$5 \times 2 = 10$

$5 \times 4 = 20$

$5 \times 8 = 40$

Ask children to describe what they notice about the equations.



Dòng nǎo jīn:

'Some children are calculating how many legs there are on six spiders.'

Ben:

'I know that six eights are forty-eight'

6 × 8 = 48

Akisha:

'Five times eight is forty, so six times eight is forty plus eight.'

6 × 8 = 5 × 8 + 8

= 40 + 8

6 × 8 = 48

Tasha:

6 × 4 = 24

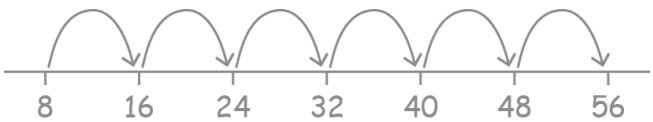
double 24 = double 20 + double 4

= 40 + 4

= 44

so 6 × 8 = 44

Elliot:



6 × 8 = 56

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		Mandy: 6 × 2 = (12) double 12 = 10 + 10 + 2 + 2 = 20 + 4 6 × 4 = 24 double 24 = 20 + 20 + 4 + 4 = 48 6 × 8 = (48) • *'Who is right?'* • *'Correct the ones that are wrong.'* • *'Whose method do you like best? Why?'*
4:13	Provide children with some practice related to links between the two, four and eight times tables.	Missing-number problems: *'Fill in the missing numbers.'* $1 \times 8 = 2 \times 4 = \square \times 2$ $2 \times 8 = \square \times 4 = 8 \times 2$ $3 \times 8 = \square \times 4 = \square \times 2$ $4 \times 8 = 8 \times \square = 16 \times \square$ $\square \times 8 = 10 \times 4 = \square \times 2$ $3 \times 2 = \square$ $3 \times 4 = \square$ $3 \times 8 = \square$ $2 \times 5 = \square$ $4 \times 5 = \square$ $8 \times 5 = \square$ $7 \times 2 = \square$ $7 \times 4 = \square$ $7 \times 8 = \square$ $20 \times 2 = 40$ $20 \times 4 = \square$ $20 \times 8 = 160$

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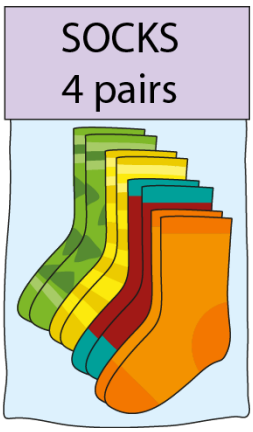
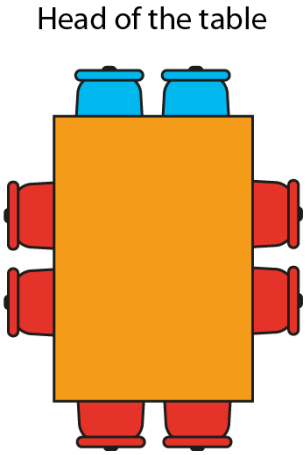
Contextual problems:

- *'A baker makes twenty-four cakes. How many boxes would he need if:'*
  - *'one box holds <u>two</u> cakes'*
  - *'one box holds <u>four</u> cakes'*
  - *'one box holds <u>eight</u> cakes?'*

- *'A shop sells socks in packs of four pairs.'*



  - *'How many <u>socks</u> are there in <u>one</u> pack'*
  - *'How many <u>pairs</u> are there in <u>three</u> packs?'*
  - *'How many <u>socks</u> are there in <u>three</u> packs?'*
  - *'If I buy thirty-two socks, how many packs is this? How many <u>pairs</u> of socks is this?'*
- *'Guests at a party sit at tables like this.*



There are eight guests at each table. There are four pairs of guests at each table. There are two guests at the head of each table.'

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'Fill in the missing numbers in the chart.'

Number of tables	Number of guests (× 8)	Number of pairs of guests (× 4)	Number of guests at head of tables (× 2)
1	8	4	2
2	16		4
3	24	12	
4		16	8
	40		10

Dòng nǎo jīn:

'For each equation, fill in the missing numbers so that the number in the first box is as large as possible and the equation is true.'

17 = ☐ × 2 + ☐

17 = ☐ × 4 + ☐

16 = ☐ × 8 + ☐

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Teaching point 5:

Divisibility rules can be used to find out whether a given number is divisible (to give a whole number) by two, four or eight.

Steps in learning

	Guidance	Representations
5:1	Knowing if a dividend is divisible exactly by a divisor (to give a whole number), without having to do a full calculation, is a useful skill. In this teaching point, children explore and apply the divisibility rules for divisors of two, four and eight. Note that, since children are working within the context of integers, throughout this teaching point the statement 'can be divided by' implies 'gives a whole number when it is divided by'. Children have already learned the divisibility rule for a divisor of two: '***If the ones digit of a number is even, the number can be divided by two.'*** Recap this now and practise sorting some numbers up to 999, according to whether they can be divided by two or not.	*'Sort these numbers according to whether they can be divided by two or not.'* 8 | 162 | 12 | 72 | 31 | 34 | 480 | 223 | 99 | 5 | 700 **_Can_ be divided by 2** | **_Can't_ be divided by 2**
5:2	Now explore the divisibility rule for division by four. Using the familiar representations and method from *Teaching point 2*, reinforce the link between the two and four times tables, focusing on facts in both times tables that have the same product. Ask children to describe connections between the times tables. Useful connections include: • Products in the two and four times tables are all even numbers. • Products in the four times table are also in the two times table.	

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<td>

The product of an even number and two is a product in the four times table.
___ times four is equal to double-___ times two.
___ times two is equal to half-of-___ times four.

Now start to consider, as a class, what divisibility rule could work for four. Ask 'Are all even numbers divisible by four?' Then explore even numbers up to 20 to discover that every other even number is divisible by four.

Explore the even numbers up to 20 again, halving twice and working towards the generalisation: 'If a number is divisible by four, halving it twice gives a whole number.'

Then ask children what they notice about the value of half of a multiple of four. Ask whether they can use what they notice to write a more efficient divisibility rule for four: 'If a number is divisible by four, halving it gives an even number.'
</td>
<td>
Deriving a divisibility rule for four:
<table>
<tr><th>Even numbers</th><th>Halve</th><th>Halve again</th></tr>
<tr><td>2</td><td>1</td><td></td></tr>
<tr><td>4</td><td>2</td><td>1</td></tr>
<tr><td>6</td><td>3</td><td></td></tr>
<tr><td>8</td><td>4</td><td>2</td></tr>
<tr><td>10</td><td>5</td><td></td></tr>
<tr><td>12</td><td>6</td><td>3</td></tr>
<tr><td>14</td><td>7</td><td></td></tr>
<tr><td>16</td><td>8</td><td>4</td></tr>
<tr><td>18</td><td>9</td><td></td></tr>
<tr><td>20</td><td>10</td><td>5</td></tr>
<tr><td>Not a divisibility rule for 4.</td><td>'If a number is divisible by four, halving it gives an even number.'</td><td>'If a number is divisible by four, halving it twice gives a whole number.'</td></tr>
</table>
</td>
</tr>
<tr>
<td>5:3</td>
<td colspan="2">Taking the more efficient divisibility rule from step 5:2, now consider whether we can use the rule to work out if a given three-digit number is divisible by four. Provide some given multiplication facts and try out the rule to see if it works. Then see if we can use just a part of the number to test for divisibility by four, beginning by just looking at the ones digit. Work towards the following generalisation: 'For numbers with more than two digits: if the final two digits are divisible by four then the number is divisible by four.'

Explore why this rule works by partitioning, for example, 156 into 100 and 56. Children have already learnt that 100 can be composed of four twenty-fives (Spine 1: Number, Addition and Subtraction, segment 1.17); you can use a hundred grid, or other familiar representation to remind children of this. Then extend to examples with a hundreds digit greater than one.</td>
</tr>
</table>

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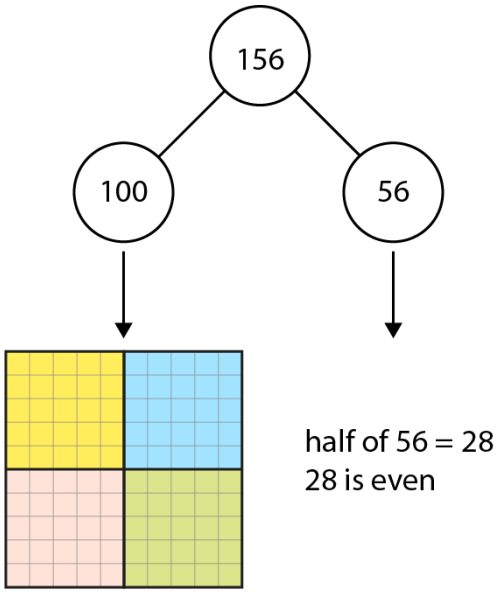
Testing the divisibility rules:

Known multiple of 4	Halve the number; is it even?	Halve the ones digit; is it even?	Halve the final two digits; is it even?
156	78	3	28
288	144	4	44
	Divisibility rule works, but we have to halve the three-digit number	*Not a divisibility rule for four.*	***'For numbers with more than two digits: if the final two digits are divisible by four then the number is divisible by four.'***

Exploring the divisibility rule:



divisible by 4 **divisible by 4**

5:4

Provide children with some practice relating to divisibility by four, for example:

- *'A factory makes ping-pong balls and sells them in packs of four.'*
  - *'Can 568 ping-pong balls be divided into whole packs of four?'*
  - *'Can 998 packs of ping-pong balls be divided into whole packs of four?'*
  - Dòng nǎo jīn: *'How many ping-pong balls will be left over?'*

'Circle the numbers that are divisible by four.'

48	214	96
148	224	106
948	234	996

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5:5

Once children have a clear understanding of the divisibility rule for four, and can confidently apply it, move on to divisibility by eight. You can use a table to explore the divisibility rule for eight, keeping to single- and two-digit numbers. Encourage children to reason that since eight is double four, we can use this to derive the divisibility rule for eight: we simply halve again (halve twice) before seeing if the result is an even number.

Deriving a divisibility rule for eight:

Number	Multiple of 4?	Multiple of 8?	Half	Half again
4	✓		2	1
8	✓	✓	4	2
12	✓		6	3
16	✓	✓	8	4
20	✓		10	5
24	✓	✓	12	6
28	✓		14	7
32	✓	✓	16	8
			'If a number is divisible by <u>four</u>, halving it gives an even number. '	***'If a number is divisible by <u>eight</u>, halving it <u>twice</u> gives an even number.'***

5:6

In a similar way to step *5:3*, you can explore whether the rule can be applied to just the final two digits in numbers with three or more digits. Work with children to reason why the rule can't be simplified for three-digit numbers: since 100 (and multiples of 100) is not divisible by eight (to give a whole number), the hundreds cannot be partitioned off and treated separately.

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5:7	Provide children with some practice relating to divisibility by eight, for example:
	• *'A factory makes pencils and sells them in packs of eight.'*
	  • *'Can 160 pencils be divided into whole packs of eight?'*
	  • *'Can 260 packs of pencils be divided into whole packs of eight?'*
	  • Dòng nǎo jīn*: 'How many pencils will be left over?'*
	Watch for children who apply the divisibility rule for eight to only the final two-digits of a three-digit number.
		'Circle the numbers that are divisible by eight.'
		48 52 96
		148 152 104
		248 252 114

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