MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Fractions

3.6 Multiplying whole numbers and fractions

Teacher guide | Year 4

Teaching point 1:

Repeated addition of proper and improper fractions can be expressed as multiplication of a fraction by a whole number.

Teaching point 2:

Repeated addition of a mixed number can be expressed as multiplication of a mixed number by a whole number.

Teaching point 3:

Finding a unit fraction of a quantity can be expressed as a multiplication of a whole number by a fraction.

Teaching point 4:

A non-unit fraction of a quantity can be calculated by first finding a unit fraction of that quantity.

Teaching point 5:

If the size of a non-unit fraction is known, the size of the unit fraction and then the size of the whole can be found.

Overview of learning

In this segment children will:

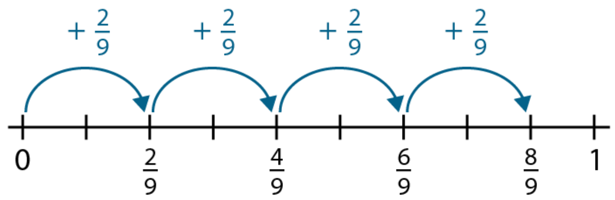
- learn that multiplying a whole number by a fraction can be thought of in two different ways: repeated addition and scaling
- learn why a whole number multiplied by a proper fraction results in a smaller number
- use their understanding of unit fractions, non-unit fractions and wholes, and interchange between each of these (for example, from a unit fraction to a whole or from a whole to a non-unit fraction)
- work with fractions as numbers and with fractions as operators.

This segment builds on children's previous learning from *Spine 2: Multiplication and Division* that repeated addition of whole numbers can be rewritten as multiplication. They will take this previous learning with whole numbers and apply it to fractional quantities. This is the first time children will encounter the multiplication sign within their work with fractions.

This segment will focus on two main structures through which to consider multiplication of a whole number by a fraction. The first is through interpreting multiplication by a whole number as repeated addition, which will draw heavily on the unitising work they have done throughout this spine. The second is to think about multiplication by a proper fraction as 'scaling down'.

Structure 1: Repeated addition

The calculation $\mathbf{4}\times\frac{2}{9}$ (or $\frac{2}{9}\times\mathbf{4}$) uses the knowledge that four groups of $\frac{2}{9}$ is $\frac{8}{9}$.



In this approach, the fraction ($\frac{2}{9}$) is the *quantity* being worked with, and the whole number (4) is the *operator* (*'There are four of that quantity.'*).

Structure 2: Scaling

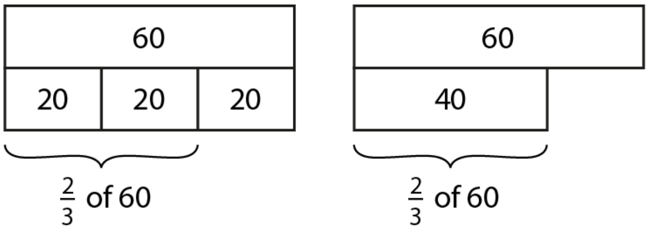
The second structure interprets multiplication simply as 'of' (finding a fraction *of* a quantity). For example,

- $\frac{2}{3}\times\mathbf{60}$ (or $\mathbf{60}\times\frac{2}{3}$)
- $\frac{1}{3}$ *of* 60 is 20

so

- $\frac{2}{3}$ must be 40

In time, children should see that 40 is the result of a 'scaling down by $\frac{2}{3}$ of 60'. This is effectively 'shrinking' the 60 to two-thirds of its original size, rather than seeing it as partitioning 60 into three parts and taking two of those parts. When we multiply by a number greater than one, we are scaling *up*. When we multiply by a number less than one, we are scaling *down*.

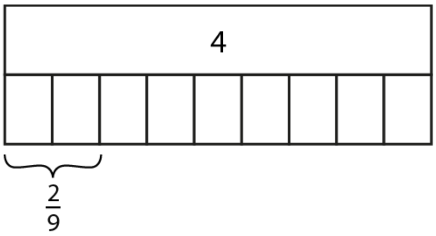


Partitioning Scaling

In this approach, the whole number (60) is the *quantity* I am working with, and the fraction ($\frac{2}{3}$) is the *operator* (*'I have two-thirds of that quantity.'*).

It might be helpful to think of these approaches as *quantities of fractions* (structure 1) versus *fractions of quantities* (structure 2). With time, children will become confident with interchanging between these, for example seeing that $\mathbf{4} \times \frac{\mathbf{2}}{\mathbf{9}}$ can be thought of as scaling ($\frac{2}{9}$ *of* 4) and as repeated addition (4 lots of $\frac{2}{9}$).

A full comprehension of this concept (for example, calculating $\frac{2}{9}$ of 4, as modelled below), is beyond the remit of the programme of study in primary schools.



The universal equivalence of these two approaches cannot be meaningfully taught at primary level. As a result, we consider these two models as different ways of thinking about the multiplication of a whole number and a fraction. Although we look at one example where children can easily see both models, our focus here is helping children develop the flexibility to move between the two models, and to decide which way of thinking about a specific calculation is more helpful.

Throughout this segment, children draw on prior learning from *Spine 2: Multiplication and Division*. They will need to be confident and secure in their understanding of the commutativity of multiplication to identify the easiest way to multiply a fraction and whole number. For example, $\mathbf{4} \times \frac{\mathbf{2}}{\mathbf{9}}$ is most easily solved by thinking of *'four lots of two-ninths'*. They therefore need to recognise that the order of the two numbers being multiplied is irrelevant. As children are making links to previous learning, they may be inclined to consider and use the language of 'factor times factor equals product'. A factor must be an integer, i.e. a whole number, so it is not accurate to describe a fraction as a factor. Children will also be introduced to the use of the multiplication symbol to replace 'of' in given statements. Note when 'of' is used, e.g. $\frac{1}{4}$ *of* 12 = 3, this is a statement and not an equation.

Remember that children have not yet learnt about equivalent fractions, so even when answers to calculations can be simplified (e.g. $\mathbf{4} \times \frac{\mathbf{1}}{\mathbf{8}} = \frac{\mathbf{4}}{\mathbf{8}} = \frac{\mathbf{1}}{\mathbf{2}}$), leave them in their un-simplified form ($\mathbf{4} \times \frac{\mathbf{1}}{\mathbf{8}} = \frac{\mathbf{4}}{\mathbf{8}}$). This will also help maintain the focus on the structure of, in this example, four one-eighths. Children have already learnt how to convert mixed numbers to improper fractions and vice versa; this skill is integrated into this segment.

As with the previous segment, there are three key models that will help children to understand the concepts that are introduced. These are the area model, number line and bar model. It is vital that children use all three of these models, as they allow structures to be seen in different ways. It is also important for children to use concrete manipulatives to support them as they learn these new concepts; rods are particularly helpful in building comprehension of the concept of scaling.

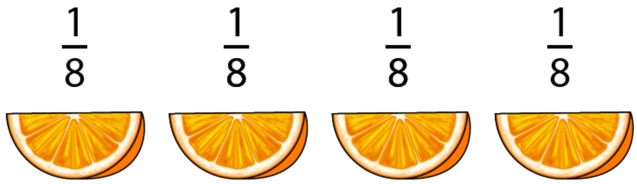
An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

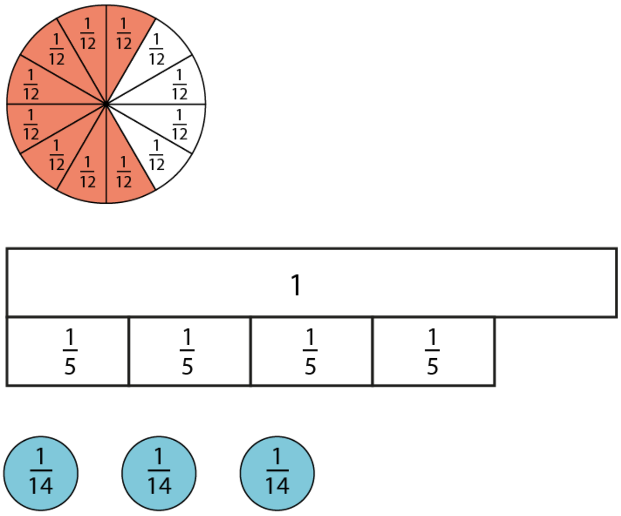
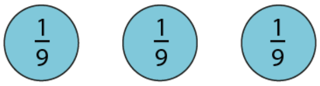
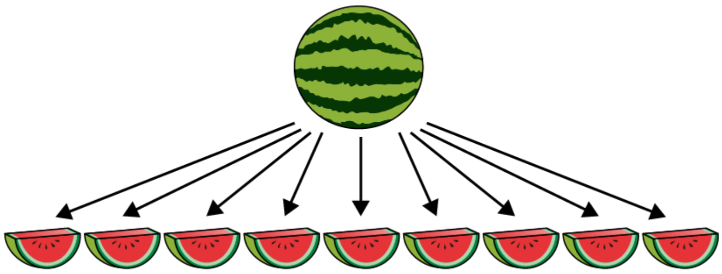
Teaching point 1:

Repeated addition of proper and improper fractions can be expressed as multiplication of a fraction by a whole number.

Steps in learning

	Guidance	Representations
1:1	The key learning in this initial teaching point is revisiting the link children made between repeated addition and multiplication in *Spine 2: Multiplication and Division*, and applying this learning to the repeated addition of fractions. Initially, children should focus on how equations can be written in different forms, without being distracted by the product. Begin by returning to a previous image from *Spine 2*. Show a representation of two, four times, for example in the context of pairs of socks, as provided opposite. Ask children to write addition and multiplication expressions to represent this image. Ask what each number represents. *'The "4" represents the number of groups, and the "2" represents the size of each group.'* Review the commutativity of multiplication. Both 4×2 and 2×4 can be used to accurately represent this image. It is not the position of the factors that is important but the meaning of each factor (i.e. that the '4', wherever it is positioned, refers to the number of groups, and the '2', wherever it is positioned, refers to the size of each group). Show that these expressions are equal to each other by joining them in an equation. By now children should be very familiar with equations in a variety of forms, and understand that this is showing that the three expressions are all equivalent. Repeat with other contexts and allow	 $2 + 2 + 2 + 2$ 4×2 2×4 $2 + 2 + 2 + 2 = 4 \times 2 = 2 \times 4$

	children plenty of opportunity to review these key concepts (the equivalence of repeated addition and multiplication, and the commutativity of multiplication) before moving on to look at fractional parts.	
1:2	Once children have reacquainted themselves with the link between repeated addition and multiplication, show the same concept with a repeated fractional part. Again, to allow children to focus on this key step of learning, do not highlight the product. Instead, emphasise the link between repeated addition and multiplication. Starting with a unit fraction as the fractional part will allow more children to access the concept. Use a familiar context that children have worked with in previous segments, to help them establish links to prior learning. Show four segments of oranges each labelled '$\frac{1}{8}$', as shown opposite, and ask children to write addition and multiplication expressions to represent the image. Ask what each number represents within each expression. *'The "$\frac{1}{8}$" represents the size of each part (an eighth of an orange). The "4" represents the number of parts.'* The position of the numbers within the multiplication expression is irrelevant. Some children will have independently joined the three expressions in an equation. Show this now and discuss, so that all children become comfortable with this equation.	 $4\times\frac{1}{8}$ $\frac{1}{8}\times 4$ $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=4\times\frac{1}{8}=\frac{1}{8}\times 4$

1:3	Provide children with further examples, such as those provided opposite, and allow them to practise writing multiplication and addition equations where the same unit fraction is repeatedly added.	
1:4	Now move toward the abstract and challenge children with missing-number problems, without pictorial representations alongside the fractions. Note that in the examples opposite, the fractional part of the multiplicative expression is sometimes placed before the multiplication symbol, and sometimes after. Where children need additional support, you can encourage them to draw unitising/addition counters to support their learning, as shown in the first example opposite.	 $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\square\times\frac{1}{9}$ $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\square\times\frac{1}{9}$ $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=5\times\square$ $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{9}\times\square$ $\square=6\times\frac{1}{8}$ $\square=\frac{1}{8}\times 5$
1:5	Once children have mastered this concept, expose them to repeatedly added non-unit fractions. As they are still not required to calculate the solution, this is no more difficult than the previous steps of learning. By introducing it separately, any misconceptions can be addressed as a class, and previous learning applied to find the solution. Again, use familiar	Dividing a whole:  *'What fraction of the whole watermelon is each part (slice)?'*

contexts to help children access this more readily. Be aware that it may be difficult for some children to understand that multiple objects they can see actually represent a fraction of a whole quantity.

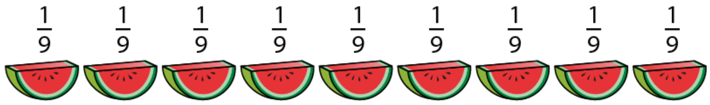
Start with a whole object, such as a watermelon. Split the object into a number of parts and discuss what each of the parts represents. This builds on previous learning. Adapt the stem sentence first encountered in segment *3.2 Unit fractions: identifying, representing and comparing*: ***'The whole has been divided into ___ equal parts, and ___ of these parts is ___.'***

Once it has been established what each part represents, write addition and multiplication expressions, as in the previous step of learning.

The image can then be manipulated so that more than one fractional part is shown in each group. Keeping the unit fraction labelled on each part and then renaming the group will help children to understand that these multiple images represent a fraction of a whole.

Write addition and multiplication expressions as before, continuing to link the expressions to the structure:

- *'What does the "$\frac{3}{9}$" represent?'*
- *'What does the "3" represent?'*

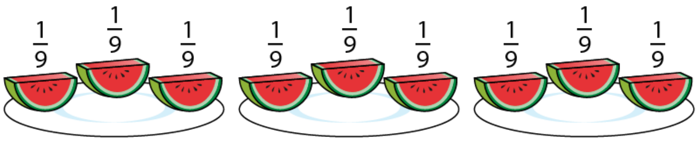


$$\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=9\times\frac{1}{9}$$

- *'The whole has been divided into nine equal parts, and one of these parts is $\frac{1}{9}$.'*

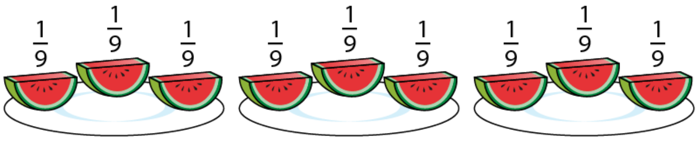
Grouping parts of a whole:

- *'The parts (slices) are arranged into three groups.'*



'What fraction of the whole does each group represent?'

- *'Write addition and mulitiplcation expressions to match the image.'*

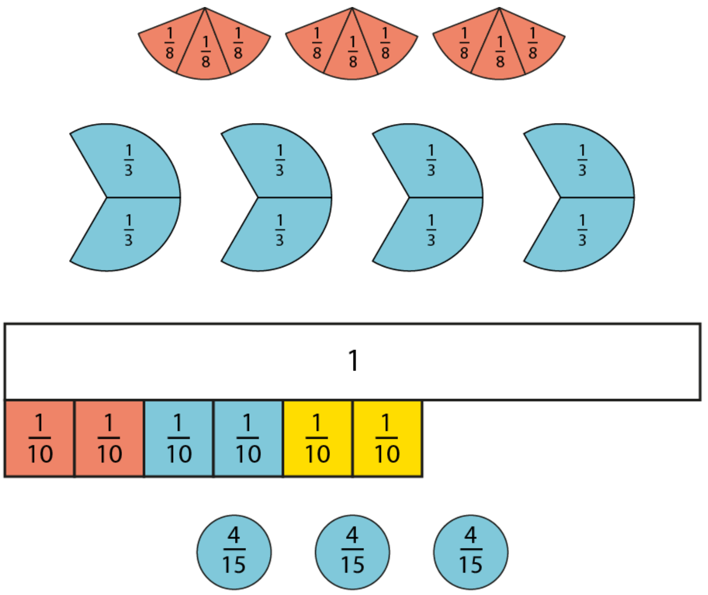


- $\frac{3}{9}+\frac{3}{9}+\frac{3}{9}$
- $3\times\frac{3}{9}$
- $\frac{3}{9}\times3$

- *'What does the "$\frac{3}{9}$" represent?'*
- *'What does the "3" represent?'*

1:6

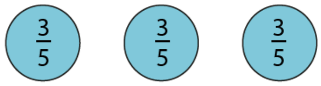
Provide further opportunities for children to write addition and multiplication expressions from given representations or practical examples. Note that in the examples you provide, the whole should be split into a number of pieces so that it can be grouped without any remainders. As children are not finding the solution at this stage, it does not matter if the pictorial representations you provide would result in answers greater than one and written as mixed numbers or improper fractions.



1:7

Once children have a good grasp of this concept, offer more abstract examples without pictorial support, as you did for unit fractions.

If needed, children can draw their own unitising counters for reinforcement, as shown in the first example opposite.



$\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}=\square\times\frac{3}{5}$

$\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}=\mathbf{6}\times\square$

$\square=\mathbf{3}\times\frac{2}{7}$

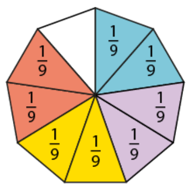
$\square=\frac{8}{7}\times\mathbf{4}$

1:8

The next stage of learning within this teaching point, is to calculate the answers to multiplication of fractions within one whole. By this point, all children should be confident in their understanding of what an abstract expression, such as $\mathbf{4}\times\frac{2}{9}$ or $\frac{2}{9}\times\mathbf{4}$,

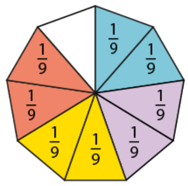
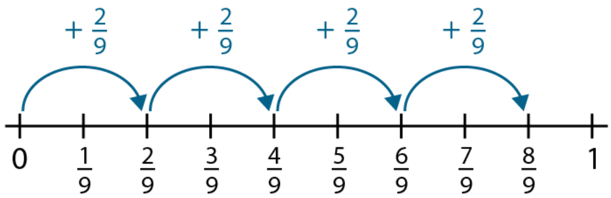
represents. If they can draw a picture or write a repeated addition expression for this calculation, then they will be able to understand the concept of finding the solution.

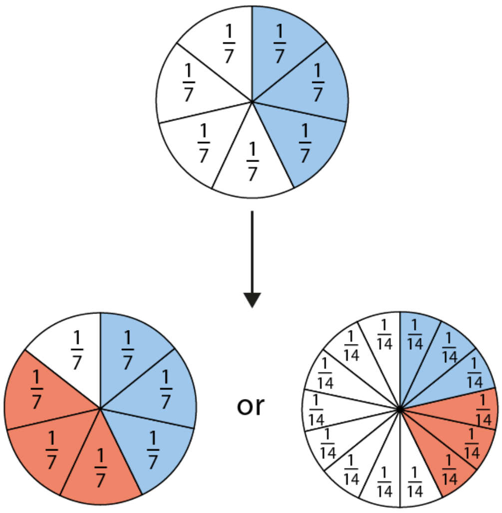
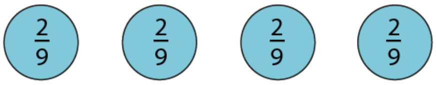
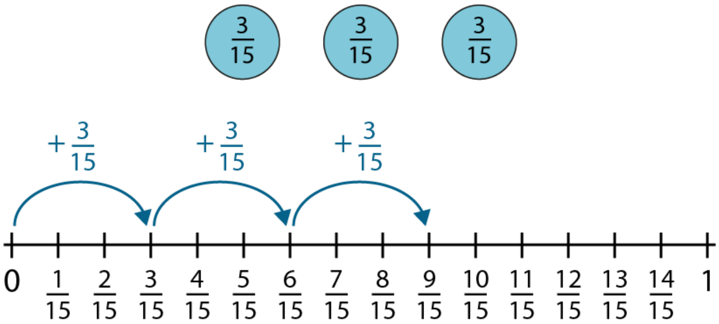
Start by presenting an image, such as



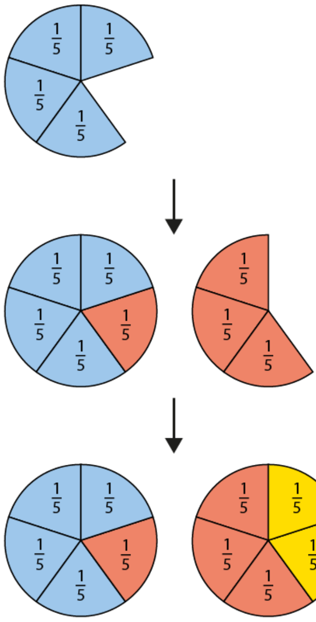
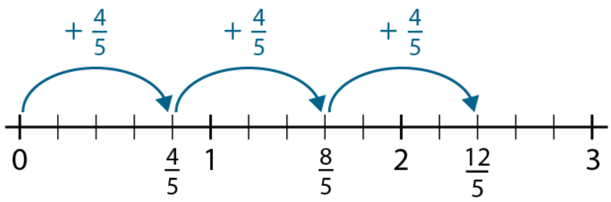
- $\frac{2}{9}+\frac{2}{9}+\frac{2}{9}+\frac{2}{9}$
- $\mathbf{4}\times\frac{2}{9}$
- $\frac{2}{9}\times\mathbf{4}$
- *'Four lots of* $\frac{2}{9}$ *is equal to* $\frac{8}{9}$*.'*

	the area model on the previous page, where it is easy to identify that four lots of $\frac{2}{9}$ is equivalent to $\frac{8}{9}$. Show the children the corresponding addition and multiplication expressions, and conclude with the children that these expressions are just different ways of representing the same thing. Ask the children to look very carefully at the image again, and this time they should discuss with a partner how many ninths in total there are in four lots of $\frac{2}{9}$. Focusing firmly on the image, rather than on 'solving the equation', will help the children to see that the answer must be $\frac{8}{9}$. If they start focusing on the equations only at this stage, there is more of a danger they will write things like $4\times\frac{2}{9}=\frac{8}{36}$, so only work verbally at this stage and keep their focus on the area model. To reiterate this, you could say something like: *'Remember, we are thinking about how many ninths there are in four lots of* $\frac{2}{9}$*. Let's look at this image to see how many ninths this is in total.'* Discuss with the class why the total must be $\frac{8}{9}$, still discussing verbally, rather than writing an answer to the equations. Use a stem sentence, as shown below and on the previous page. ***'___ lot(s) of ___ is equal to ___.'***	

1:9	Next, introduce a zero–one number line, marked in units of $\frac{1}{9}$. Representing the repeated addition on a number line while skip counting, is another way to make sense of this calculation. It will also help avoid the misconception that we add the denominator. Count together as a class: *'Two-ninths, four-ninths, six-ninths, eight-ninths.'* You can use the associated slide to support this action: *3.6 Representations*, slide *8*. Discuss the links between the area model and the number line. These are just two different possible ways of proving that four lots of $\frac{2}{9}$ is $\frac{8}{9}$. When children have completed these steps, return to the initial expressions and insert the solution. It is important for children to see what happens to the denominator and the numerator of the fraction in the question and in the solution. Links should be made back to the addition of fractions work they did in segment *3.4 Adding and subtracting within one whole*, where the denominator also didn't change when fractions with the same denominator were added. Spend some time looking at the three equations, and discuss any questions the children have about them. Children may form a generalisation at this stage that: ***'The numerator of the fraction is multiplied by the whole number and the denominator remains the same.'*** More practice examples should be completed so that all children reach this conclusion for themselves, rather than relying only on procedural steps.	  $\frac{2}{9}+\frac{2}{9}+\frac{2}{9}+\frac{2}{9}=\frac{8}{9}$ $4\times\frac{2}{9}=\frac{8}{9}$ $\frac{2}{9}\times4=\frac{8}{9}$

1:10	Exposing children to incorrect calculations will allow them to identify and explain what procedural mistakes have been made, and therefore why conceptually they do not work. In general, asking children to explain why something is incorrect allows better judgement of their level of understanding. The most likely error is that they multiply both the numerator and the denominator by the whole number. Look at the image in *Example 1*. Ask children to imagine what it would look like if $\frac{3}{7}$ was multiplied by two. Emphasise that the denominator doesn't change because the unit has not changed; several lots of the same units are combined. The images showing $\frac{6}{7}$ and $\frac{6}{14}$ should reinforce this.	Example 1: $\frac{3}{7} \times 2 = \frac{6}{14}$ ✗  Example 2: $\frac{3}{10} \times 2 = \frac{6}{20}$ ✗
1:11	Look at more of these calculations. While the area model can be a useful way to represent the calculations, it is difficult to draw quickly. A more efficient way is to draw repeated unitising counters. Display what this would look like for the examples in step *1:9*, and link it to the three equations shown opposite.	 $\frac{2}{9}+\frac{2}{9}+\frac{2}{9}+\frac{2}{9}=\frac{8}{9}$ $4 \times \frac{2}{9}=\frac{8}{9}$ $\frac{2}{9} \times 4=\frac{8}{9}$
1:12	Repeat this process with a different example, but this time using unitising counters rather than the area model as the main model. Also link the counters to repeated jumps of $\frac{3}{15}$ on a number line. Notice how the fraction is given before the whole number in the first expression opposite ($\frac{3}{15} \times 3$). Check that the children are still able to identify this as *'three lots of* $\frac{3}{15}$' rather	$\frac{3}{15} \times 3$ 

	than '$\frac{3}{15}$ *lots of three*'. While the latter is mathematically correct, as explained in the overview, it is a much harder concept for children of this age to understand. Link to the commutative multiplication equation and the repeated addition equation. Use the stem sentence from step *1:8*: ***'___ lot(s) of ___ is equal to ___.'***	$\frac{3}{15} \times \mathbf{3} = \frac{9}{15}$ $\mathbf{3} \times \frac{3}{15} = \frac{9}{15}$ $\frac{3}{15} + \frac{3}{15} + \frac{3}{15} = \frac{9}{15}$ *'Three lots of* $\frac{3}{15}$ *is equal to* $\frac{9}{15}$*.'*
1:13	Give children practice, both converting to repeated addition expressions and solving the calculation. They need to be comfortable working with the fraction as either the first or the second number in the multiplication expression. To further extend their understanding, introduce dòng nǎo jīn problems like the one opposite.	$\frac{3}{20} \times \mathbf{4} =$ $\quad$ $\mathbf{6} \times \frac{2}{14} =$ $\mathbf{3} \times \frac{5}{18} =$ $\quad$ $\frac{4}{11} \times \mathbf{2} =$ Dòng nǎo jīn: *'How many ways can you complete this equation?'* $\frac{24}{25} = \square \times \frac{\square}{25}$
1:14	The next step is to solve calculations where the solution is greater than one (e.g. $\mathbf{3} \times \frac{4}{5}$). This is approached in exactly the same way as the previous calculations, but this time the answer will be an improper fraction. This improper fraction answer can then be converted to a mixed number. At this stage, children may procedurally understand that the numerator of the fraction is multiplied by the whole number. It is fine if they utilise this conjecture when they are exposed to a question such as $\mathbf{3} \times \frac{4}{5}$, but don't let them rely solely on this procedure at this stage. The whole class should explore the solution, returning to the number line and pictorial representations to make links to repeated addition of the fraction.	$\mathbf{3} \times \frac{4}{5}$ Repeated addition: $\frac{4}{5} + \frac{4}{5} + \frac{4}{5}$

	Counting in non-unit fractional steps, i.e. *'four-fifths, eight-fifths, twelve-fifths'* using the number line alongside the area model, naturally reaches a solution that is an improper fraction. Both the area model and the number line will support children in seeing the equivalence of $\frac{12}{5}$ to $2\frac{2}{5}$. However, as the children have had lots of practice relating mixed numbers to improper fractions in segment *3.5 Working across one whole: improper fractions and mixed numbers*, they should already be confident making this conversion. Link back to the commutative multiplication expression; approach $\frac{4}{5}\times 3$ in the same way as $3\times\frac{4}{5}$. Conclude by showing how it can also be represented with repeated addition with counters.	Models:   Commutativity: $3\times\frac{4}{5}=\frac{12}{5}=2\frac{2}{5}$ $\frac{4}{5}\times 3=\frac{12}{5}=2\frac{2}{5}$ 
1:15	As discussed, while there are some benefits to using a circular area model, notably the obvious completion of each 'whole', there are also disadvantages, such as its being hard to draw accurately and independently. Instead, repeated addition of unitising counters will be the main model used. This is a more abstract model, in that the individual unit fractions are no longer visible, but it still acts as a scaffold by exposing the repeated addition within the calculation. It is also a representation you or the children	

can quickly and easily draw to support their understanding.

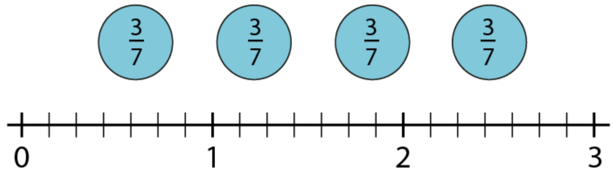
Present another calculation (e.g. $\frac{3}{7}\times 4$), and ask children to draw out repeated groups/unitising counters to represent it. Present the number line opposite and count in non-unit fractional steps along the number line, i.e. *'Three-sevenths, six-sevenths, nine-sevenths, twelve-sevenths'.* Then complete the equation.

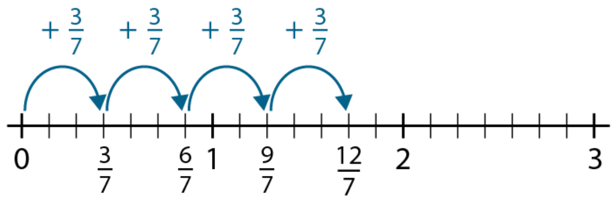
$$\frac{3}{7}\times 4=\frac{12}{7}$$

It is important to discuss with the children that although $\frac{12}{7}$ is a correct answer to this calculation, they already know how to convert $\frac{12}{7}$ to a mixed number from their work in segment *3.5*, so they can apply that learning here. The number line also provides a pictorial aid to demonstrate the equivalence between $\frac{12}{7}$ and $1\frac{5}{7}$.

As before, link to the commutative multiplication expression. $4\times\frac{3}{7}$ can be calculated in exactly the same way we have calculated $\frac{3}{7}\times 4$.

$\frac{3}{7}\times 4$





$\frac{3}{7}\times 4=\frac{12}{7}=1\frac{5}{7}$

$4\times\frac{3}{7}=\frac{12}{7}=1\frac{5}{7}$

1:16 Share some completed calculations with the class, such as the examples opposite. Look at the patterns within each calculation:

- The denominator remains the same throughout.
- The numerator is multiplied by the whole number to produce the improper fraction.
- Where there is an improper fraction it can be changed to a mixed number.

We can generalise that: ***'To multiply a fraction and a whole number, we multiply the numerator by the whole number and keep the denominator the same.'*** However, as mentioned

$4\times\frac{2}{9}=\frac{8}{9}$

$\frac{3}{15}\times 3=\frac{9}{15}$

$3\times\frac{4}{5}=\frac{12}{5}=2\frac{2}{5}$

$\frac{3}{7}\times 4=\frac{12}{7}=1\frac{5}{7}$

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<td>previously, it is important that children do not rely solely on applying this rule and maintain a solid understanding of why this is the case.</td>
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Once children are confident multiplying a fraction with a whole number that results in an improper fraction, progress to multiplying improper fractions by whole numbers. If children have a solid understanding of the previous concepts, this will not require much new teaching. All that changes is the size of the numerator in relation to the denominator. For example, finding the solution to $\mathbf{3}\times\frac{4}{5}$, is conceptually no different than solving $\mathbf{3}\times\frac{6}{5}$. At this stage, do not show the improper fraction as a mixed number as this will be discussed separately in the next teaching point.

Commence by showing children the equation $\mathbf{3}\times\frac{6}{5}$ and ask them to draw repeated addition counters to represent this.

Discuss that we are taking the same approach as before; the improper fraction is being added to itself the number of times expressed by the whole number. Because an improper fraction is greater than one, this will result in a solution greater than one. Furthermore, as each of the repeatedly added parts is greater than one, the solution will be greater than value of the whole number, in this case greater than three.

As before, show the different forms of the equation expressed as multiplication and repeated addition, and then use a number line and pictorial representations to help children understand and solve the equation.

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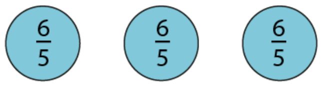
$\mathbf{3}\times\frac{6}{5}$

3 lots of $\frac{6}{5}$

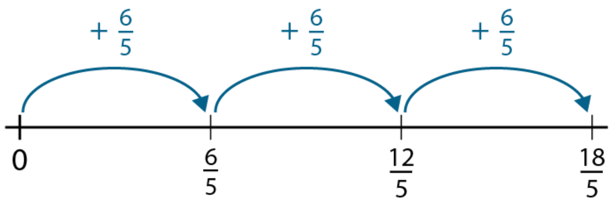
$\frac{6}{5}+\frac{6}{5}+\frac{6}{5}$



$\mathbf{3}\times\frac{6}{5}$

$\frac{6}{5}\times\mathbf{3}$

$\frac{6}{5}+\frac{6}{5}+\frac{6}{5}$



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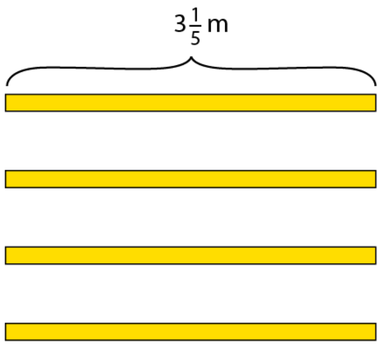
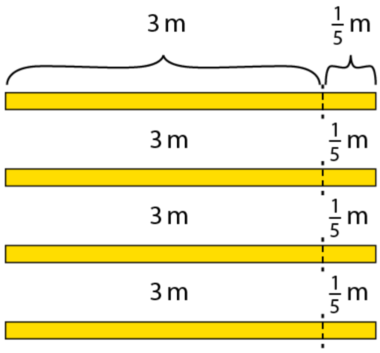
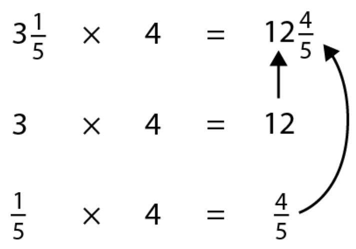
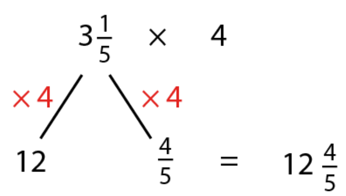
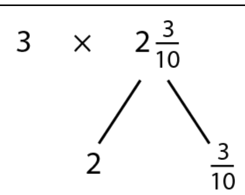
	Skip count along on the number line: *'Six-fifths, twelve-fifths, eighteen-fifths.'* Summarise that $3 \times \frac{6}{5} = \frac{18}{5}$ Once you have found a solution as a class, discuss links to the calculations discussed in step *1:16*, and how this calculation can be solved without pictorial support. Some children may be able to make these links immediately, while some may need to practise with further calculations before this is fully understood.	
1:18	Finally, give children varied practice with these types of calculations. Children should consolidate and further deepen their understanding through questions such as the dòng nǎo jīn problems provided opposite.	Word problems: • *'My family eats about* $\frac{2}{3}$ *of a box of cereal each week. How many boxes of cereal do we eat in four weeks?'* • *'The distance from home to my school is* $\frac{3}{4}$ *km. I walk to school and home again five days a week. How far do I walk each week?'* Solving equations: $\frac{5}{8} \times 4 =$ $7 \times \frac{3}{10} =$ $\frac{4}{3} \times 8 =$ $\frac{2}{9} \times 3 =$ Dòng nǎo jīn: *'Fill in the missing numbers.'* $1\frac{1}{8} = \square \times \frac{3}{8}$ $\frac{\square}{4} \times \square = 3\frac{3}{4}$

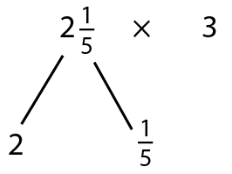
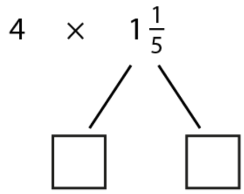
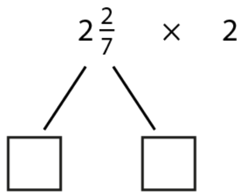
Teaching point 2:

Repeated addition of a mixed number can be expressed as multiplication of a mixed number by a whole number.

Steps in learning

	Guidance	Representations
2:1	Open this teaching point by presenting children with a real-life context (see opposite) that involves repeated addition of a mixed number (e.g. $\mathbf{3}\frac{1}{5}+\mathbf{3}\frac{1}{5}+\mathbf{3}\frac{1}{5}+\mathbf{3}\frac{1}{5}$). Begin by representing the length of ribbon that is required to make one cake as one continuous length, and then show that this is required four times. Using a length context is useful because the metre unit of the ribbon allows the partitioned parts to be easily identified and not be confused with the multiplier. Pose a question to initiate discussion, for example: *'How can we partition each* $\mathbf{3}\frac{1}{5}$ *m length of ribbon?'.* Children are likely to give different responses but if they do not suggest it, prompt them to partition $\mathbf{3}\frac{1}{5}$ into its whole number and fractional part. Show this on a model of four lengths of ribbon. Draw children's attention to the whole-number part of the metre, asking questions such as: • *'How many three-metre lengths of ribbon are there?'* • *'What calculation can we do to find this total length?'* Show the calculation *3 m × 4 = 12 m* underneath the sections of four three-metre lengths of ribbon. Then, move on to the fractional part, asking:	*'To decorate a wedding cake, a baker needs* $\mathbf{3}\frac{1}{5}$ *m of ribbon. The baker needs to make 4 wedding cakes. How much ribbon does the baker need?'*   $3\text{ m}\times 4 = 12\text{ m}$ $\frac{1}{5}\text{ m}\times\mathbf{4}=\frac{4}{5}\text{m}$ $\mathbf{12}\text{m}+\frac{4}{5}\text{m}=\mathbf{12}\frac{4}{5}\text{m}$

	• *'How many $\frac{1}{5}$ m lengths of ribbon are there?'* • *'What calculation can we do to find the total length?'* Show the calculation $\frac{1}{5}\text{m} \times \mathbf{4} = \frac{4}{5}$ underneath the sections of four $\frac{1}{5}$ m lengths of ribbon. Finally, before adding the products of the whole and fractional parts together, ask: *'Now, how can we calculate the total length of ribbon required to decorate the cake?'*. Show the calculation: $\mathbf{12}\text{m} + \frac{4}{5}\text{m} = \mathbf{12}\frac{4}{5}\text{m}$	
2:2	Once children are secure finding the solution to a given problem using pictorial support, explore how we can represent this symbolically. These steps could be presented in a variety of ways, as shown in the three examples opposite. Children must understand that both the fractional *and* whole number parts of the mixed number must be multiplied by the multiplier. This is the distributive law of multiplication, and children have already met this in multiplication of whole numbers (e.g. $27 \times 3 = 20 \times 3 + 7 \times 3$). The same principle applies here. Children may find some of these methods of presenting the steps easier to understand than others. Showing them alongside each other and asking, *'What is the same? What is different?'* will give children the opportunity to choose which representation they prefer.	Example 1: $3\frac{1}{5} \times 4 = \underbrace{3 \times 4}_{12} + \underbrace{\frac{1}{5} \times 4}_{\frac{4}{5}} = 12\frac{4}{5}$ Example 2:  Example 3: 
2:3	Give children varied practice while gradually reducing the scaffolding, including examples where the mixed number is given either before or after the multiplication sign. Use low numbers so that calculation difficulty is	

	not a barrier to understanding. The mixed number can have a unit or non-unit fractional part, but at this stage, keep the product of the fractional part less than one. Children may begin to solve these calculations mentally without the intermediate steps shown, but if they start to make mistakes, encourage them to return to using the jottings to help them better understand the structures.	$3 \times 2 = \square$ $3 \times \frac{3}{10} = \square$ $\square + \square = \square$  $2 \times 3 = \square$ $\frac{1}{5} \times 3 = \square$ $\square + \square = \square$   $3 \times 2\frac{1}{6} =$ $5\frac{2}{7} \times 3 =$
2:4	At this point, take time to address mistakes and/or misconceptions that arose during practice. Possible mistakes or misconceptions may include:	Incorrect equations: $3\frac{1}{5} \times 2 = 6\frac{1}{5}$ ✗ $6\frac{2}{9} \times 3 = 18\frac{6}{27}$ ✗

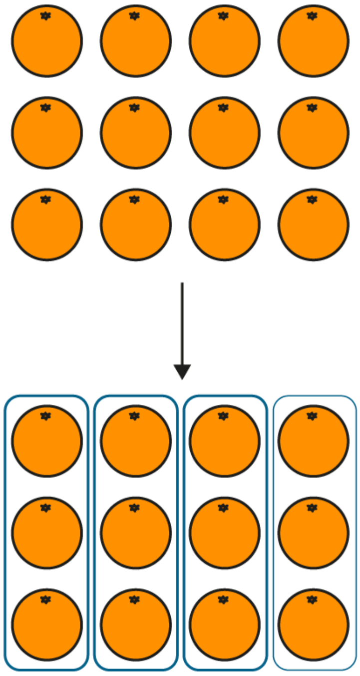
	• only multiplying the whole number by the multiplier • multiplying both the numerator and the denominator of the fractional part by the multiplier. To reduce the likelihood of the first error, the arrows that show the partitioning of the mixed number, as per the first example in step *2:3*, provide a useful prompt. For the second error, return to some of the images from *Teaching point 1*, such as step *1:10*, to remind children why only the numerator changes. Address both errors head-on with the class, using incorrect examples such as those opposite. Ask children to explain why they are incorrect and then model how to solve them correctly.	
2:5	Move onto calculations where the product of the fractional part and the multiplier will be greater than one. Children should approach these questions in the same way as the previous questions. They might initially leave their solutions in the non-conventional format of a mixed number with an improper fractional part. Remind them that the numerator should be less than the denominator within a mixed number. For example, $4\times3\frac{2}{5}=12\frac{8}{5}$, $12\frac{8}{5}$ is structurally correct, but mathematical convention requires us to change the final answer to a conventional mixed number: $4\times3\frac{2}{5}=12\frac{8}{5}$ $=13\frac{3}{5}$ Provide plenty of opportunities for practice, including contextual questions and questions involving measurement (such as the example opposite).	$4\times1\frac{2}{5}$ ☐ × ☐ = ☐ ☐ × ☐ = ☐ ☐ + ☐ = ☐ $2\frac{3}{4}\times5$ ☐ × ☐ = ☐ ☐ × ☐ = ☐ ☐ + ☐ = ☐ $2\times3\frac{4}{5}$ $3\frac{3}{4}\times2$ Measurement context: *'One bucket holds* $3\frac{3}{4}$ *litres of water. How much water can five buckets hold?'*

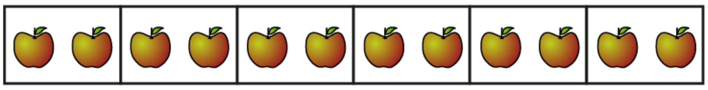
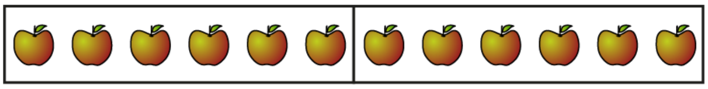
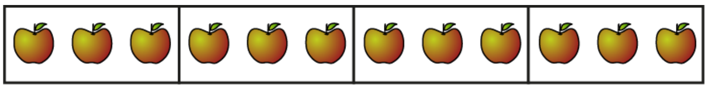
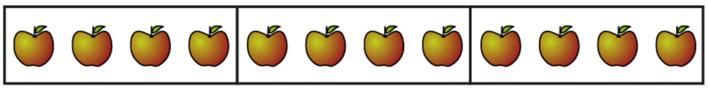
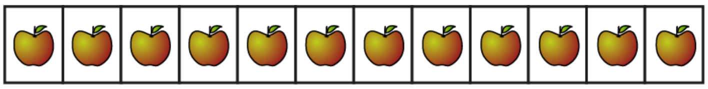
	To make the measurement context as meaningful and realistic as possible, limit fractional parts to $\frac{1}{10}$, $\frac{1}{5}$, $\frac{1}{4}$ and $\frac{1}{2}$ and multiples of these (e.g. $\frac{3}{4}$ and $\frac{2}{5}$). It is rare to encounter $2\frac{1}{7}$ kilometre, for example.	
2:6	Finally, present children with questions that are difficult to perform mentally. The concepts and structures needed to understand and solve this problem are no different to the previous questions, but most children will need a written method.	Word problem: *'The length of a day on Planet Zog is 8 hours.* *The length of a day on Planet Zig is approximately* $243\frac{1}{8}$ *times the length of a day on Planet Zog.* *In hours, what is the length of a day on Planet Zig?'* • Step 1 $243\frac{1}{8}\times 8=$ • Step 2 $\begin{array}{cccc} & 2 & 4 & 3 \\ \times & & & 8 \\ \hline 1 & 9 & 4 & 4 \\ \hline {\scriptstyle 1} & {\scriptstyle 3} & {\scriptstyle 2} & \end{array}$ • Step 3 $\frac{1}{8}\times 8=\frac{8}{8}=1$ • Step 4 $1944+1=1945$ • *'The length of a day on Planet Zig is 1945 hours.'*

Teaching point 3:

Finding a unit fraction of a quantity can be expressed as a multiplication of a whole number by a fraction.

Steps in learning

	Guidance	Representations
3:1	Up until this point, children have been exposed to multiplying a fraction by a whole number as repeated addition of the fraction. The next two teaching points still focus on multiplication of a whole number by a fraction, but conceptually explain this as scaling down the whole number by the value of the fraction, or finding a fraction of a quantity. This is initially introduced in this teaching point using unit fractions, and then with non-unit fractions in the next teaching point. In segments *3.2 Unit fractions: identifying, representing and comparing* and *3.3 Non-unit fractions: identifying, representing and comparing*, children gained experience circling or highlighting unit and non-unit quantities of an amount, for example *'Circle three-quarters of the oranges.'* To do this, they relied on a stem sentence, which, when applied to this example, would be: *'The whole is divided into four equal parts and we have three of them.'* Therefore, to an extent the children already 'know' that $\frac{3}{4}$ of 12 is 9, but they have learnt this as 'showing $\frac{3}{4}$' rather than calculating $\frac{3}{4}$ of 12. In this step, we make the link between this and the multiplication of a fraction and a whole number. Start with a brief recap of the link between the number of equal parts in a whole and the fraction each part is of the whole. Show the class a bar split	Circling a quantity: *'Circle three-quarters of the oranges.'*  • *'The whole is divided into four equal parts and we have three of them.'*

	into a number of equal parts, and use the following stem sentence to identify the fraction that each part is of the whole. • ***'The whole is divided into ___ equal parts.'*** • ***'Each part is ___ of the whole.'*** At this stage, it is important for children to understand that *each* of the parts represents the unit fraction of the whole and not just the first part. Labelling every part will allow children to cement this understanding. Repeat for different numbers of equal parts. The children should pick this up quickly, as both the language and concept will be familiar to them.	Parts of a whole: $\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$
3:2	Now start making the link between finding $\frac{1}{6}$ of 12 *with* a pictorial representation to finding $\frac{1}{6}$ of 12, *without* a pictorial method. First, return to exploring different ways to share objects into equal parts. Give children 12 apples (either in front of the class or in pairs working with counters). Place the group of apples in a row and ask the children to split the group into equal parts. Keep the apples in a row as much as possible, rather than clustered in groups or shown in arrays. This is to allow a natural progression to using the bar model. Allow children to suggest the different ways 12 can be divided into equal parts, and record their suggestions pictorially on the board. When all possibilities have been shared, discuss what is the same and what is different	 *'The whole is divided into **six** equal parts.'*  *'The whole is divided into **two** equal parts.'*  *'The whole is divided into **four** equal parts.'*  *'The whole is divided into **three** equal parts.'*  *'The whole is divided into **twelve** equal parts.'*

in each representation. Pose questions such as:

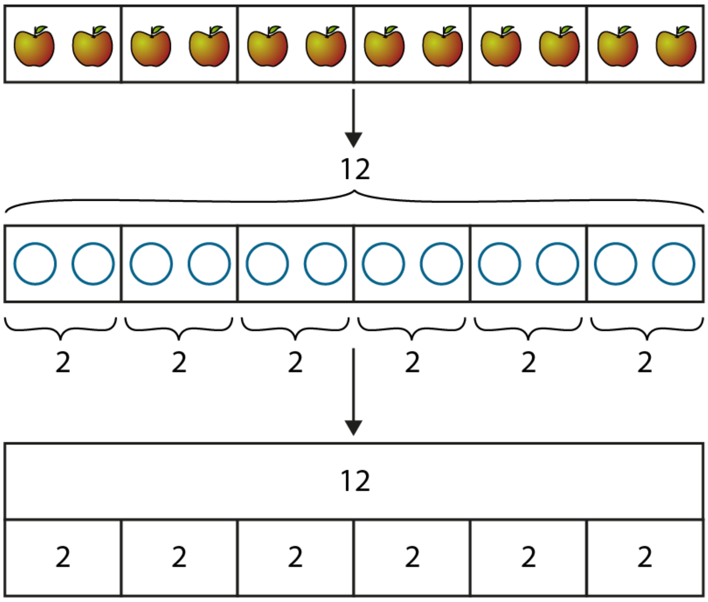
- *'What is the whole number of apples in each representation?'*
- *'What is the number of apples in each part?'*

As children identify these similarities and differences, label them on the representations. Label the whole in each model as 12 and then label the number of apples in each part of the model. Once each part has been labelled, remove the individual circles that represent each apple, to leave the more abstract bar model as shown opposite.

Summarise using the following stem sentence: ***'When ___ apples are split into ___ equal parts, there are ___ apples in each part.'***

Repeat this for all of the different ways of splitting the 12 apples into equal groups. At this stage you don't need to make the link to division – the children can simply describe what they see in the images.

Note: The language of apples is still used here. If you move too quickly to work in the abstract, children can get confused between whether each part is two or whether each part is $\frac{1}{6}$. Of course, each part is *both* (two apples *and* $\frac{1}{6}$ of the total number of apples), but it is important to help children understand this distinction.



*'When **twelve** apples are split into **six** equal parts, there are **two** apples in each part.'*

3:3

Next, link these models to fractions of a quantity with the children using sentences such as '$\frac{1}{6}$ ***of*** *12 = 2.'* Note that this is a sentence and not an equation as it uses 'of' instead of the '×' symbol. Look again at the bar model from the previous step, but this time bring in the fractional language from step *3:1*, asking

- *'How many parts has the whole been split into?'*
- *'What fraction of the whole does each part represent?'*

and using the stem sentences:

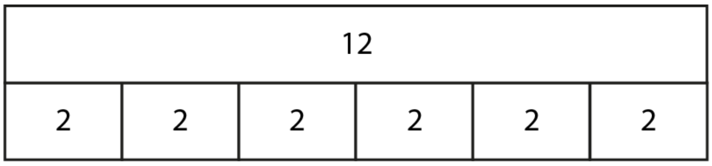
- ***'The whole is divided into ___ equal parts.'***
- ***'Each part is ___ of the whole.'***

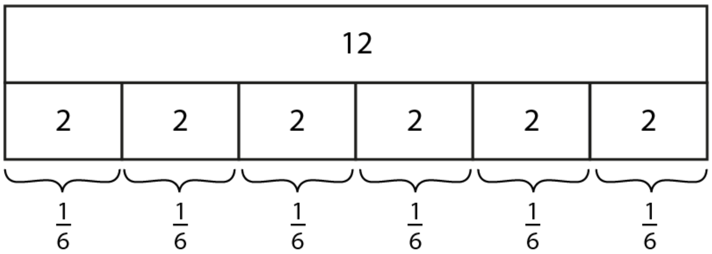
Once children have discussed and reached their conclusion, label the representations accordingly. As before, it is important for children to understand that *each* of the parts represents this unit fraction of the whole, and not just the first part.

When this is has been fully grasped, move on to identifying what *quantity* this fraction of the whole represents. It is important that children are able to distinguish between the number of items (here, the number of apples) in each whole, and the fraction of the whole that this represents (here, $\frac{1}{6}$ of the total apples). Giving the children a scaffold in the form of a stem sentence will help to support this, as will continuing to use the unit of 'apples':

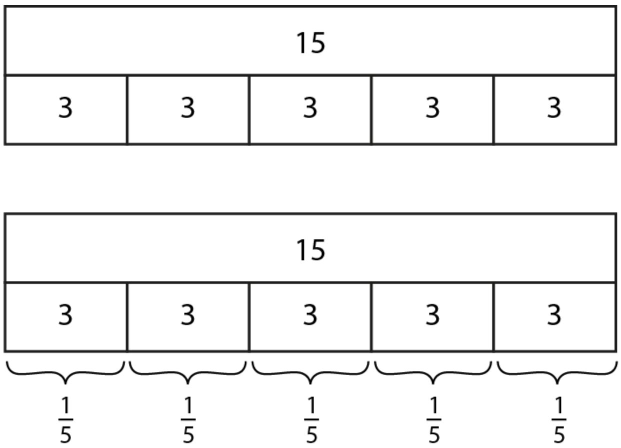
'Each part is $\frac{1}{\square}$ ***of the whole;***

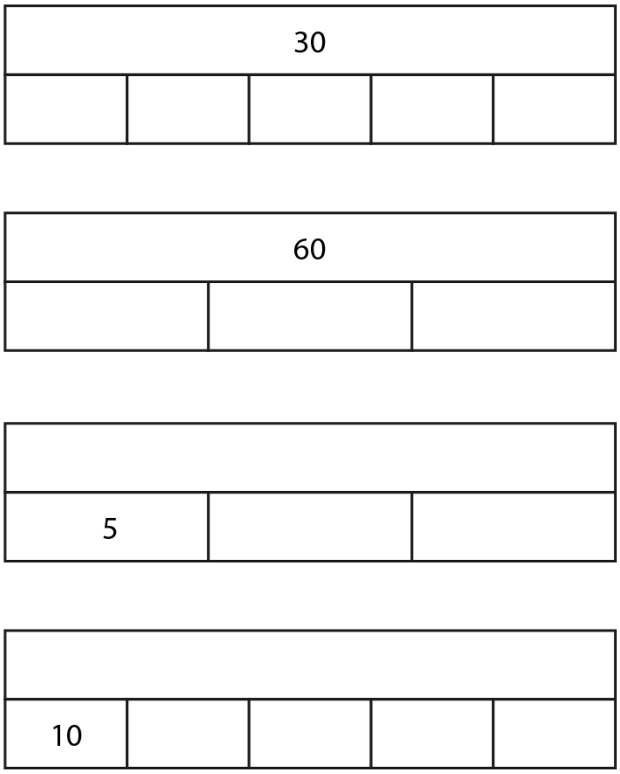
$\frac{1}{\square}$ ***of ___ apples is ___ apples.'***

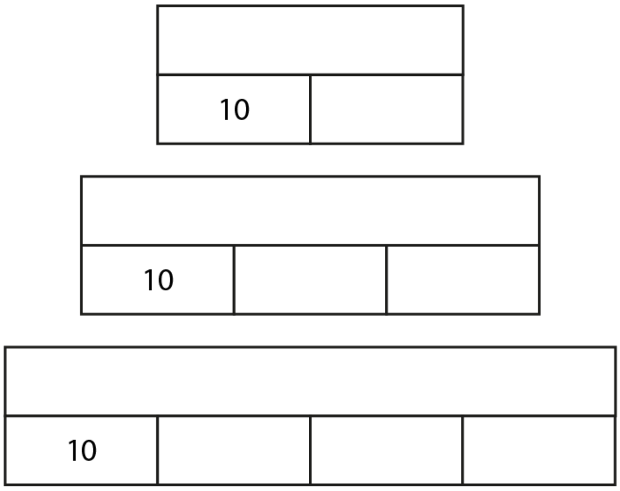
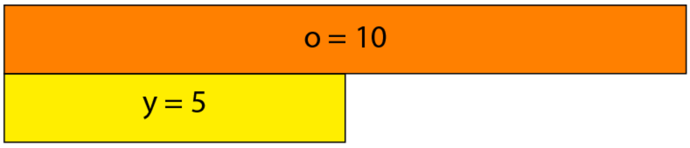
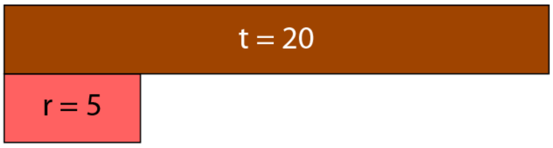
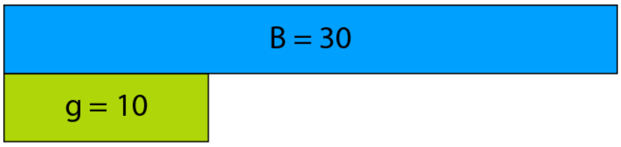




'Each part is $\frac{1}{6}$ of the whole; $\frac{1}{6}$ of twelve apples is two apples.'

	If using real apples, interchange between looking at the model on the board with working with the real apples at the front of the class. Arrange the apples into equal groups to mirror the image on the board (as shown above), saying: *'My apples are divided into six equal groups. Can someone please hand me* $\frac{1}{6}$ *of the apples?'* When you are presented with two apples, encourage the class to repeat the last part of the stem sentence, for example: *'*$\frac{1}{6}$ *of twelve apples is two apples.'* Take time to emphasise that *any* two apples make up $\frac{1}{6}$ of the whole, so encourage the children to take any pair, not always the first pair in the row. (At this stage it makes for a closer link to the sixths if the children take one of the 'pre-arranged' groups rather than two individual apples from different places in the line – although this, of course, is still correct.) Focus on some alternative ways of breaking up the 12 apples, making the link between the practical activity and a visual image of the bar model on the board. Continue to repeat this until the children are confident with the language.	
3:4	Now consider some bar models with a different whole and different numbers of equal parts (where the whole and value of each part is already marked), as shown opposite. Repeat the same stem sentence to express the quantity that each fractional part is of the whole: ***'Each part is*** $\frac{1}{\square}$ ***of the whole;*** $\frac{1}{\square}$ ***of ___ is ___.'***	

	If children find this difficult, return to the previous stem sentence in step *3:3* to help them to identify the fraction that each part represents. Then present a written statement of the fraction of the quantity.	*'Each part is $\frac{1}{5}$ of the whole;* *$\frac{1}{5}$ of 15 is 3.'*
3:5	To develop this idea further, present children with models that are only partially labelled, for example, with only the whole or the number of parts given. For these examples, children should complete the missing information (using simple mental calculations) and then produce statements from the completed bar models. During this process, make links between division and finding a fraction of a quantity to strengthen children's understanding of the concept. Providing children with scaffolds to complete for each model will ensure the focus remains on unit fractions of a quantity. • ***'___ is divided into ___ equal parts; each part is $\frac{1}{\square}$ of the whole.'*** • ***'$\frac{1}{\square}$ of ___ = ___'*** To deepen understanding, present dòng nǎo jīn problems, where a calculation is completed in different ways and an appropriate model for each statement given. For scaffolding, provide children with the beginning of a sequence of solutions and appropriate correlating models, and ask them to continue the sequence.	 • *'Thirty is divided into five equal parts; each part is $\frac{1}{5}$ of the whole.'* • *'$\frac{1}{5}$ of 30 = 6.'* Dòng nǎo jīn: $\frac{1}{\square}$ of $\square$ = 10 $\frac{1}{2}$ of $\square$ = 10 $\frac{1}{3}$ of $\square$ = 10 $\frac{1}{4}$ of $\square$ = 10

		
3:6	Once children can confidently describe fractions of a quantity in this way, progress to seeing multiplication as scaling. Approach this by leading on directly from the previous step by using the bar model. This time, do not show all of the parts. Instead describe the relationship of the two bars to each other. When children can see these relationships, they are more likely to understand how multiplying a whole number by a fraction makes the whole number smaller. Cuisenaire® rods are a particularly useful resource for enabling children to see the multiplicative relationship of two rods relative to each other. Provide a stem sentence to help children describe the relationship between the rods. For example: ***'___ of ___ = ___'*** ***'___ lots of ___ = ___'*** Provide children with opportunities to describe similar relationships with different numbers of parts, such as in the examples opposite. You could use values *not* based on the white rod being '1', but allow them to check the relationships, if required.	 *'$\frac{1}{2}$ of 10 = 5'* *'2 lots of 5 = 10.'*  

3:7

Now introduce the concept of replacing 'of' with the multiplication sign.

Begin with the concept that children are familiar with. Explain that:

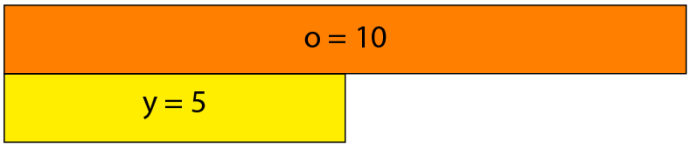
- *'lots of'* in a sentence such as 2 lots of 5 = 10 can be replaced with the multiplication sign to make $2 \times 5 = 10$.
- *'of'* in the sentence $\frac{1}{2}$ of 10 = 5 can also be replaced with the multiplication sign to give $\frac{1}{2} \times 10 = 5$.

At this point, introduce the term *'scaling'* and explain that we can think of this as scaling the number down. Ten has been scaled-down by $\frac{1}{2}$ to make five.

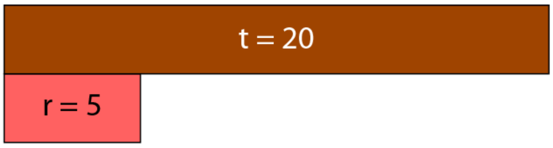
Return to images from the previous step and replace each instance of 'lots of' and 'of' with the multiplication sign. Also provide children with the opportunity to work in the other direction and replace the multiplication sign with 'lots of' or 'of'. To consolidate this, give children practice in matching the equivalent expressions.

At this stage, it is also important that children understand that when a whole number is multiplied by a fraction, the whole number becomes smaller. Show children the calculations where they have replaced 'of' with '×' and ask questions that will allow children to make an appropriate generalisation, such as: ***'When a whole number is multiplied by a unit fraction, it makes the whole number smaller.'***

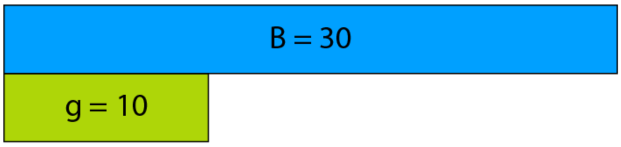
Moving between 'of' and multiplication symbol:



- 2 lots of 5 = 10
- $2 \times 5 = 10$
- $\frac{1}{2}$ of 10 = 5
- $\frac{1}{2} \times 10 = 5$



- $\frac{1}{4}$ of 20 = 5
- $\frac{1}{4} \times 20 = 5$



- $\frac{1}{3}$ of 30 = 10
- $\frac{1}{3} \times 30 = 10$

Once the children are secure in their understanding of this generalisation, provide missing-symbol comparison statements. To keep the focus on applying an understanding of the relationships, choose numbers where the multiplication of the fraction and the whole number can't be easily calculated.

To deepen understanding, introduce dòng nǎo jīn problems.

Missing-symbol comparison statements:

'Fill in the missing symbols (<, > or =) to make each statement true.'

$27 \times \frac{1}{5} \bigcirc 27$

$50 \bigcirc \frac{1}{4} \times 50$

Dòng nǎo jīn:

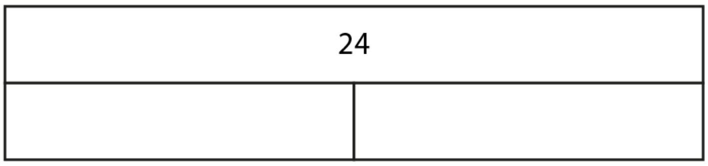
'Complete the missing parts on the bar models and then describe the relationships between the solutions to the following calculations.'

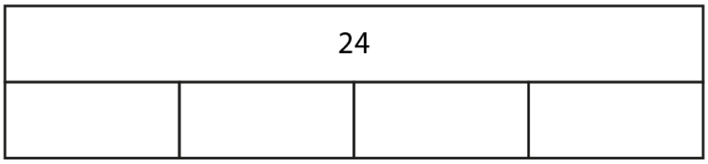
1 of 24

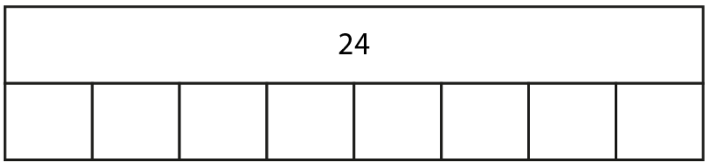
$\frac{1}{2}$ of 24

$\frac{1}{4}$ of 24

$\frac{1}{8}$ of 24







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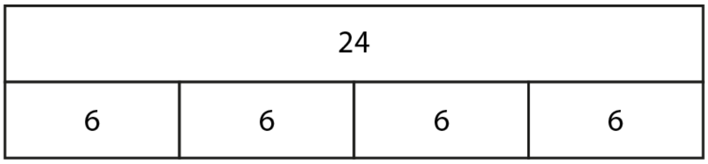
<td>3:8</td>
<td>When children are able to confidently rewrite multiplication calculations using both ‘of’ and ‘×’, look at calculations such as $\frac{1}{4} \times 24 = ?$

Establish that the steps to solve these questions are the same as before. Work through this first calculation, and demonstrate the construction of the bar model alongside the abstract equation.
• Draw two touching bars.
• Write the whole in the top.
• Split the bottom bar into the correct number of equal parts.
• Write a division equation to calculate the size of each part.
• Write the multiplication equation.

Look at the resulting equation and notice how, as with division by a whole number, multiplication of a whole number by a fraction results in a smaller number. The initial whole number is scaled-down.

Work through several examples, making sure that children see calculations presented both ways, e.g. $\frac{1}{4} \times 24$ and $24 \times \frac{1}{4}$. Remind children that multiplication is commutative and these are exactly the same calculations. $\frac{1}{4} \times 24$ might more obviously lend itself to being seen as ‘$\frac{1}{4}$ of 24’, but $24 \times \frac{1}{4}$ can equally be seen as ‘$\frac{1}{4}$ of 24’. At this stage, only provide calculations that can be calculated mentally, without requiring the use of short division.

At this point, children may begin to identify links to previous teaching points and notice that $\frac{1}{4} \times 24$ is both 24 ‘one-quarters’ and one-quarter of 24. This will be addressed in more detail in the next teaching point.</td>
<td>$\frac{1}{4} \times 24 =$

$24 \div 4 = 6$
So,
$\frac{1}{4} \times 24 = 6$</td>
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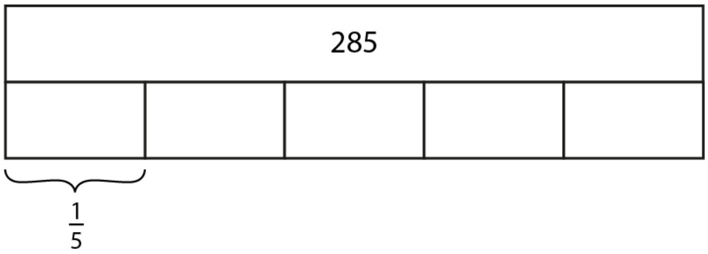
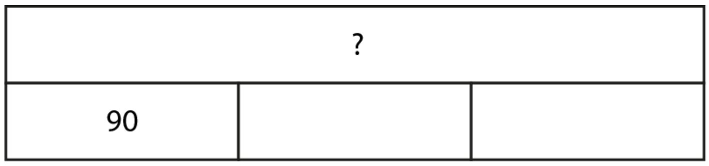
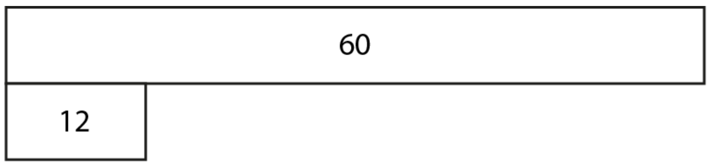
3:9	To further consolidate the equivalence of multiplication of whole numbers by fractions and division by an integer, provide children with practice, including: • matching multiplication and division equations, e.g $\frac{1}{3} \times 204$ and $204 \div 3$ • moving between forms Choose numbers where children are unlikely to be able to quickly calculate the product, as this will allow them to focus more on the equivalent ways of writing the same calculation.	Matching multiplication and division: *'Circle the calculations that are equivalent to* $\frac{1}{3} \times 204$.*'* $204 \times \frac{1}{3}$ 3×204 $204 \div 3$ $3 \div 204$ $\frac{1}{204} \times 3$ Moving between forms: *'Fill in the missing numbers to complete the equivalent calculations.'* $\frac{1}{9} \times 414 = \square \div \square$ $\frac{1}{\square} \times \square = 270 \div 4$
3:10	Provide independent practice such as that shown opposite. Children who have a deep understanding of how to move forwards and backwards between parts and wholes as a result of multiplication of whole numbers and fractions, will be able to explain how they know the solution to the dòng nǎo jīn problem opposite, without having to divide 70 by three. Children with a less secure understanding may procedurally try to find the solution and become stuck at the first stage of the calculation. Understanding that you can 'undo' a multiplication by an integer by multiplying by a unit fraction is a critical idea at secondary school level. Children will learn that $\frac{1}{5}$ is the reciprocal of five and that multiplying a number by its reciprocal always gives one. Questions similar to those in the previous step will provide further good dòng nǎo jīn problems.	*'Complete the equations.'* $\frac{1}{8} \times \mathbf{32}$ $\frac{1}{3} \times \mathbf{15}$ $\frac{1}{5} \times \mathbf{40}$ $\mathbf{21} \times \frac{1}{7}$ $\mathbf{36} \times \frac{1}{6}$ $\mathbf{28} \times \frac{1}{4}$ Dòng nǎo jīn: $70 \times \frac{1}{3} \times \square = 70$ $45 = 45 \times 5 \times \frac{\square}{\square}$

3:11	Move on to problems requiring a written method to solve the division. The concept is exactly the same as before, so encourage children to draw a bar model and write the division equation underneath. They will then need to use short division to find the value of the unit fraction.	$\frac{1}{5}$ of 285 =  $$\begin{array}{r} 0\;5\;7 \\ 5\overline{)2\;{}^{2}8\;{}^{3}5} \end{array}$$ 285 ÷ 5 = 57
3:12	Finally, provide children with contextualised problems requiring a unit fraction of a quantity to be calculated. Provide calculations that can and cannot be calculated mentally, as with the ideas given opposite. The first problem can be solved with a mental method and the second with a written method. By this stage, many children may not need to use a bar model to interpret the problems they are presented with. Children should be encouraged to move away from drawing the model, instead mentally interpreting the division calculation from the worded problem. Provide children with dòng nǎo jīn questions that give the size of a part and require them to calculate the whole. Or provide the part and whole and challenge them to find what unit fraction has been calculated. Examples of how you might approach modelling this are given opposite.	• *'$\frac{1}{5}$ of a class of 30 children can sit in each row in assembly. How many children sit in each row?'* • *'A school charity run raises £1592. Four charities each get $\frac{1}{4}$ of the money. How much does each charity receive?'* Dòng nǎo jīn: • *'There are 90 children in $\frac{1}{3}$ of the playground. How many children are in the whole playground, if there are an equal number of children in each part?'*  • *'60 children choose their favourite sport. 12 children choose swimming. What fraction of the children is this?'* 

Teaching point 4:

A non-unit fraction of a quantity can be calculated by first finding a unit fraction of that quantity.

Steps in learning

Guidance | **Representations**

4:1

Guidance

In this teaching point, build on children's understanding of finding a unit fraction of a quantity to find a non-unit fraction of a quantity. Return to using a concrete context, providing practical opportunities to reinforce the concept. For example, place 15 apples on a table and choose a child to take $\frac{1}{5}$ of the apples, asking questions to encourage them to explain their reasoning, such as:

- *'How many apples did you take?'*
- *'How did you know to take this number of apples?'*
- *'What calculation did you complete?'*

The child should explain that they divided the whole into five groups and then took one of the groups, which was three apples. Record this as a calculation on the board.

Arrange the 15 apples in defined groups of three. Ask children in the class to take incremental quantities of the apples, for example *'Take two one-fifths of the apples, take three one-fifths of the apples'*, and so forth. Count up as the children take the apples: *'One one-fifth, two one-fifths, three one-fifths'*. Saying the fraction in the form of *'x one-fifths'* will allow children to more easily make the link to the unit fraction of a quantity.

Record each calculation on the board, using the equations to express something they already know. They have seen with the physical apples that if we take $\frac{3}{5}$ of the 15 apples, we have

Representations

Representing taking $\frac{1}{5}$ of 15 apples:

$\frac{1}{5} \times 15 = 3$

Representing counting in $\frac{1}{5}$ of 15 as equations:

$\frac{1}{5} \times 15 = 3$

$\frac{2}{5} \times 15 = 6$

$\frac{3}{5} \times 15 = 9$

$\frac{4}{5} \times 15 = 12$

$\frac{5}{5} \times 15 = 15$

Bar model:

15				
3	3	3	3	3

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<tr>
<td></td>
<td>taken nine apples, which can be written as $\frac{3}{5}\times\mathbf{15}=\mathbf{9}$.

When this sequence has been completed, show the same scenario on a bar model, linking it back to the calculations previously completed. As each calculation is discussed, shade the required parts of the bar model to show how this number is represented.

Once children have a solid understanding, ask 'If I wanted to find, for example, $\frac{3}{5}$ of 15, how could I do it?'

Encourage them to explain how to calculate this, based on the activities that have been completed as a class. Then, develop more mathematical language that can be generalised to any fraction of a quantity, for example:

'To calculate $\frac{3}{5}\times\mathbf{15}=\mathbf{9}$, find $\frac{1}{5}$ of 15 and then multiply by 3.'</td>
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<td>4:2</td>
<td>Repeat the steps above using a different number of apples and a different number of parts (such as 14 apples divided into seven equal groups), again linking the practical example to the bar model. Conclude as before: 'To calculate $\frac{4}{7}\times\mathbf{14}$, find $\frac{1}{7}$ of 14, and then multiply by four.'

When the children are confident, introduce the following generalisation: 'To calculate a fraction of a quantity, find the unit fraction of the quantity. Then multiply the unit fraction by the numerator.'

Throughout both of these steps, and as you progress, continue to alternate between the language of 'of' and 'times' when reading aloud the equations.</td>
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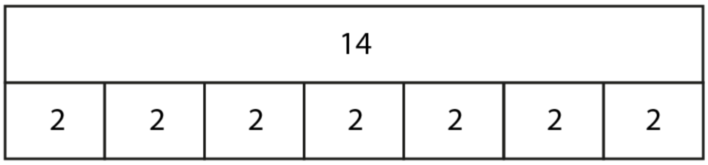

$\frac{1}{7}\times\mathbf{14}=\mathbf{2}$

$\frac{2}{7}\times\mathbf{14}=\mathbf{4}$

$\frac{3}{7}\times\mathbf{14}=\mathbf{6}$

$\frac{4}{7}\times\mathbf{14}=\mathbf{8}$</td>
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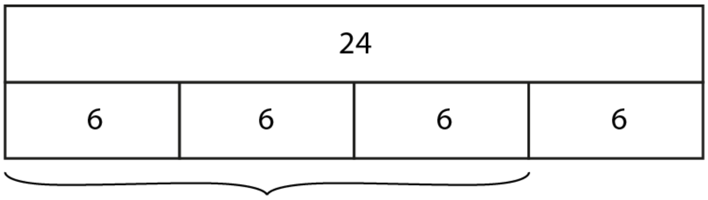
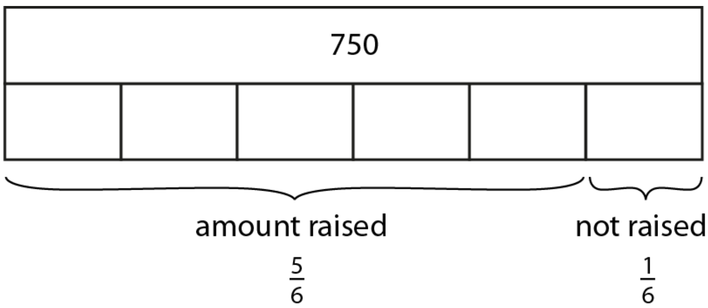
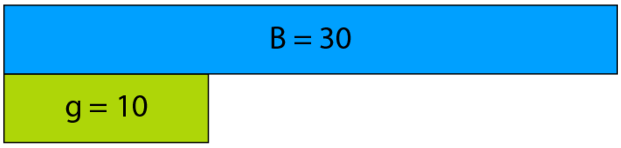
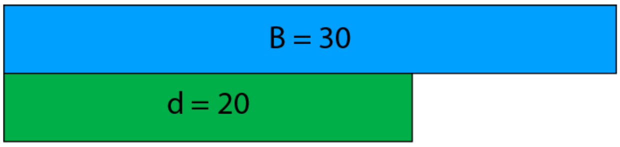
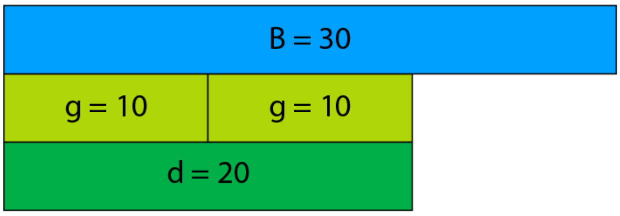
</table>

4:3	Next, proceed to multiplying a fraction by a whole number where children need to construct their own bar model. Show the class the expression $\frac{3}{4}\times\mathbf{24}$ and ask them to represent this on a bar model. Make the link between the fraction and the calculation $\frac{3}{4}\times\mathbf{24}$. The fraction literally tells us what we need to do: the whole (24) is divided into four equal parts and we want three of those parts. Discuss the bar model as a class and then break down the steps in the calculation: $24 \div 4 = 6$ $\frac{1}{4}\times\mathbf{24}=\mathbf{6}$ $\frac{3}{4}\times\mathbf{24}=\mathbf{18}$ Summarise the steps: *'To calculate* $\frac{3}{4}\times\mathbf{24}$*, find* $\frac{1}{4}$ *of 24 and then multiply by 3.'* Guide the children through several more examples. A few children may be ready to work with just the written calculations without drawing out the model, but don't rush to do this. Along with the stem sentences, the bar model is a helpful scaffold that children can quickly draw to help them when they need it.	$\frac{3}{4}\times\mathbf{24}=$  $24 \div 4 = 6$ $\frac{1}{4}\times\mathbf{24}=\mathbf{6}$ $\frac{3}{4}\times\mathbf{24}=\mathbf{18}$ *'To calculate* $\frac{3}{4}\times\mathbf{24}$*, find* $\frac{1}{4}$ *of 24 and then multiply by 3.'*
4:4	Provide independent written practice, as shown opposite. At this stage, children's understanding of the commutativity of multiplication means they can also be given questions in the form of a whole number multiplied by a fraction, e.g. $\mathbf{49}\times\frac{6}{7}$	Word problems: • *'Stan bought 15 litres of paint and used* $\frac{2}{3}$ *of it decorating his house. How much paint has he used?'* • *'My granny lives 120 km from us. We are driving to see her and are* $\frac{5}{6}$ *of the way there. How far have we driven so far?'* • *'A teacher has 28 books to mark. He has marked* $\frac{3}{7}$ *of them. How many books does he still have left to mark?'*

	The following three problems are, of course, all the same: • *'Find $\frac{6}{7}$ of 49'* • *'Calculate $\frac{6}{7}\times 49$'* • *'Calculate $49\times\frac{6}{7}$'* However, you may find that children find the first example the easiest to access as the language essentially tells them how to solve it. Take all opportunities to reinforce that these are three different ways of writing the same calculation, so that children gain confidence with all three formats. Children should consolidate and further deepen their understanding through practice with a range of questions, such as the dòng nǎo jīn problems shown opposite.	Equations: • *'Find:'* $\frac{3}{8}$ of 32 $\frac{2}{9}$ of 45 $\frac{3}{5}$ of 30 • *'Calculate:'* $\frac{2}{3}\times 21$ $36\times\frac{5}{6}$ $\frac{3}{4}\times 28$ $50\times\frac{7}{10}$ $\frac{4}{5}\times 15$ $42\times\frac{5}{7}$ Dòng nǎo jīn: • *'Explain why the following are equivalent.'* $\frac{3}{4}\times 20 = 20\div 4\times 3$ • *'I am $\frac{3}{4}$ of the way through my holiday. I have three days left. How many days have I already been on holiday for?'* • *'I have £2 of my birthday money left. I spent the other $\frac{9}{10}$ of it on a watch. How much birthday money did I have altogether?'*
4:5	Once children can consistently solve calculations where the individual steps can be completed mentally, proceed to questions that require short division and multiplication, e.g. $224\times\frac{5}{7}$. Also provide questions that require a non-unit fraction of a quantity to be calculated in differently-worded contexts, such as shown opposite. At this point, children may present different ways to find the solution. Those who have a deep understanding of the concept may find $\frac{1}{6}$ of the amount of money and then subtract this from the whole. This could be discussed with the class if it occurs, or could be presented for the class to determine if it is a valid alternative way	Equations: $\frac{3}{5}\times 175$ $\frac{2}{7}\times 154$ $315\times\frac{4}{9}$ $272\times\frac{5}{8}$ Word problem: *'A school are trying to raise £750 to send Year 5 on school camp. They have raised $\frac{5}{6}$ of the total. How much have they raised so far?'* 

	to find the solution. Using a bar model makes this easier to understand. Allow children to further explore this concept through dòng nǎo jīn problems like those opposite.	Dòng nǎo jīn: *'What is the value of the missing part?'* • *'$\frac{4}{5}$ of the runners finished the race. If 92 people finished the race, how many dropped out?'* • *'There are 314 cows on a farm. $\frac{3}{5}$ of the cows are having calves this year. How many cows are not having calves?'*
4:6	In *Teaching point 3*, children were presented with a model to show that when calculating a unit fraction of a number, a whole number becomes smaller when it is multiplied by a unit fraction. This was initially introduced as scaling and understood as a result of describing the relationships between two bars, using the stem sentence: **'___ of ___ = ___.'** *For example:* *'$\frac{1}{3}$ of 30 = 10'* This was then extended to $\frac{1}{3} \times \mathbf{30} = \mathbf{10}$, once children learnt to recognise that '*of*' and '×' mean the same thing. Allow children to re-familiarise themselves with this model. Then, introduce the non-unit fraction form of the same model and ask children to describe the relationship between the two bars. Providing an additional layer to the model may help more children to describe the relationship. They can also use the stem sentence **'___ of ___ = ___.'** Or, they may write it as an equation (e.g. $\frac{2}{3} \times \mathbf{30} = \mathbf{20}$).	Multiplying a whole number by a *unit* fraction as scaling:  *'$\frac{1}{3}$ of 30 = 10'* $\frac{1}{3} \times \mathbf{30} = \mathbf{10}$ Multiplying a whole number by a *non-unit* fraction as scaling:   *'$\frac{2}{3}$ of 30 = 20'* $\frac{2}{3} \times \mathbf{30} = \mathbf{20}$

Return to the previous generalised statement: ***'When a whole number is multiplied by a unit fraction, it makes the whole number smaller.'***

Extend this generalisation now to all proper fractions (both unit and non-unit): ***'When a whole number is multiplied by a proper fraction, it makes the whole number smaller.'***

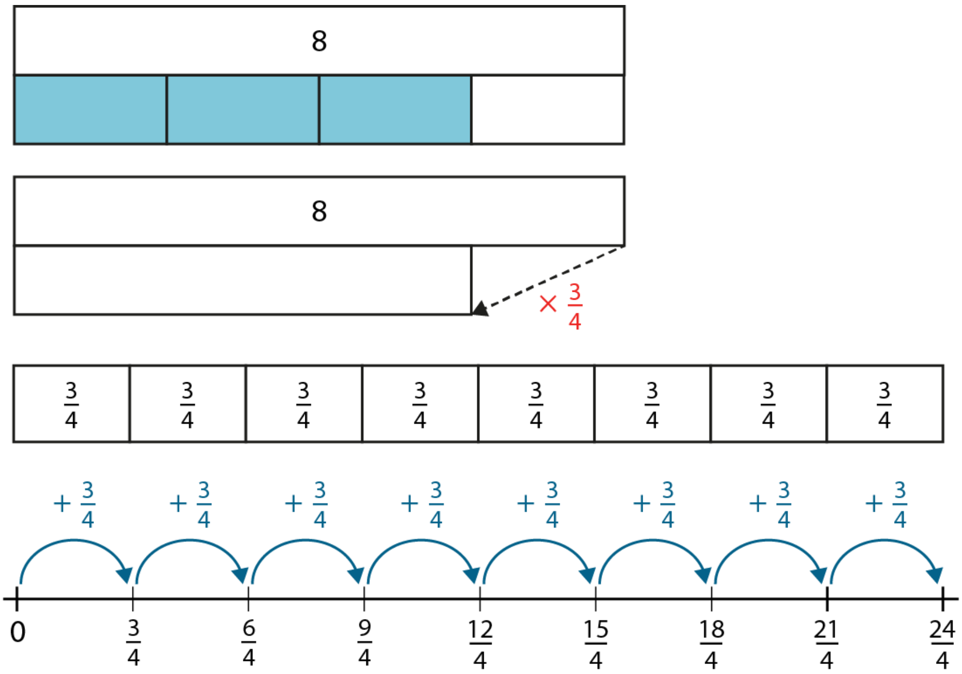
The idea that multiplying a whole number by a proper fraction is a 'scaling down' of the whole number that makes it smaller, becomes really important in secondary school. In some branches of higher mathematics, division ceases to be a concept that is used, since any division can be replaced by a multiplication (for example, dividing by four is the same as multiplying by one-quarter). The notion of multiplication as scaling may help here, since scaling can make things larger or smaller. Moving away from always thinking that multiplication makes things bigger is an important awareness to develop in preparation for secondary school.

4:7

At this stage, children have been introduced to multiplication of whole numbers with unit and non-unit fractions in the contexts of repeated addition and scaling. As mentioned in the *Overview of learning*, the level of mathematical knowledge children need in order to understand that every multiplication of a fraction and whole number can be seen and calculated in these two ways, goes beyond the primary programme of study. However, there are some examples where the numbers involved mean that children can see, through the use of visual models, that these two structurally-different approaches both lead to the same answer. The expression $\mathbf{8} \times \frac{3}{4}$ is one such calculation. This can be seen as:

- 8 lots of $\frac{3}{4}$ (repeated addition model)
- $\frac{3}{4}$ of 8 (scaling model)

Show children the calculation $\mathbf{8} \times \frac{3}{4}$ and ask them to discuss with their partner the different ways this could be interpreted and represented pictorially and in written form. Share the ideas as a class, discussing the similarities and differences of each approach. A range of models which may be useful to support this discussion are shown on the next page. Discuss which of these models, plus those generated by the children, represent 8 lots of $\frac{3}{4}$ and which represent $\frac{3}{4}$ of 8.



$\frac{3}{4} \times 8$ $\quad$ $\frac{3}{4}$ of 8

$8 \times \frac{3}{4}$ $\quad$ 8 lots of $\frac{3}{4}$

$= \frac{24}{4}$

$= 6$

4:8

When children meet multiplication of a fraction and a whole number, there are no hard-and-fast rules about which of these calculation approaches to take. You can invite them to explore different methods and then, as a class, discuss which ones work best in particular contexts. For example:

- $8 \times \frac{3}{4}$ can easily be tackled both ways.
- $7 \times \frac{3}{4}$ is probably best approached using repeated addition, as finding $\frac{1}{4}$ of 7 is not something primary-aged children will yet have solid conceptual understanding of.
- $16 \times \frac{3}{4}$ on the other hand, is probably better suited to the scaling method,

'Calculate:'

$24 \times \frac{2}{5}$ $\qquad$ $\frac{7}{8} \times 56$

$30 \times \frac{4}{6}$ $\qquad$ $\frac{2}{3} \times 13$

	as finding $\frac{1}{4}$ of 16 is easier than multiplying 3 by 16. • $\mathbf{444} \times \frac{3}{4}$ would certainly be easier to tackle with scaling – finding $\frac{1}{4}$ of 444 and then $\frac{3}{4}$ of 444. If the denominator of a fraction is a factor of the whole number, then the scaling method is generally more appealing. Using either of the following models, we are basically completing the same two steps: • In the repeated addition model, the calculation $\mathbf{7} \times \frac{3}{4}$, is essentially $7 \times 3 \div 4$ • In the scaling model, the calculation $\mathbf{16} \times \frac{3}{4}$, is also essentially $16 \div 4 \times 3$. Now provide practice, looking at each calculation in turn and considering which method is easier. To start with, it may help if you model aloud for children, for instance: *'$\mathbf{7} \times \frac{3}{4}$: should I do 7 lots of $\frac{3}{4}$ or $\frac{3}{4}$ of 7?* *I don't know what $\frac{1}{4}$ of 7 is, so I think I will do 7 lots of $\frac{3}{4}$. That is $\frac{21}{4}$. Now I just need to convert it to a mixed number.'*	

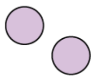
Teaching point 5:

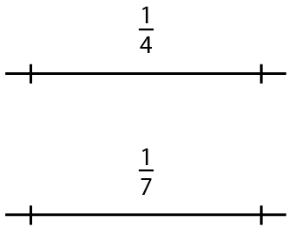
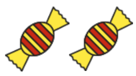
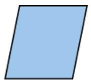
If the size of a non-unit fraction is known, the size of the unit fraction and then the size of the whole can be found.

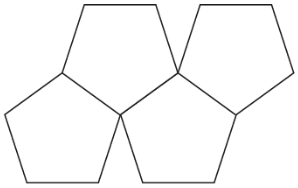
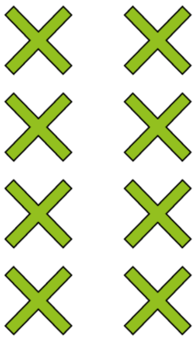
Steps in learning

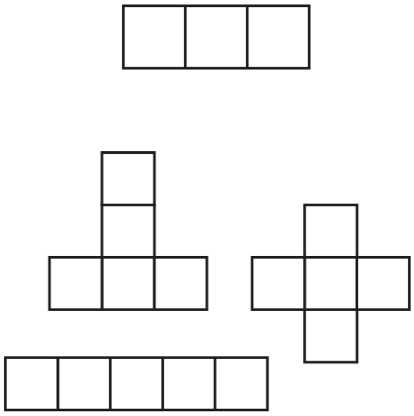
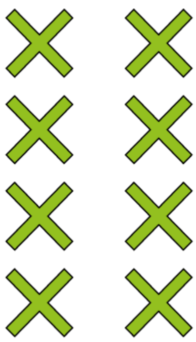
	Guidance	Representations
5:1	In segments *3.1 Preparing for fractions: the part–whole relationship* and *3.2 Unit fractions: identifying, representing and comparing*, children learnt how to move from one part to a whole: ***'If we know the size of a unit fraction, we can work out the size of the whole.'*** In this teaching point, this concept is applied to moving from a non-unit fraction to the whole and also to moving from a fraction of a quantity to calculating the whole quantity. Children need to understand this as a two-step process: • Step 1: moving from a non-unit fraction to a unit fraction • Step 2: moving from a unit fraction to the whole (this has already been covered). Begin by recapping the learning from segment *3.2, Teaching point 6* in order to ensure children are secure with moving from a unit fraction to a whole and vice versa (*step 2* above). Return to the table first introduced in segment *3.1* where children completed columns one and four. This was revisited in segment *3.2* to include all four columns, as shown in the examples opposite. Provide a table with missing information (like the one opposite), so children can quickly practise this learning once again. Pay particular attention to the third row, and check that children have drawn 16 circles rather than eight.	(see tables below)

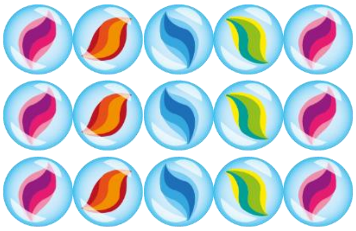
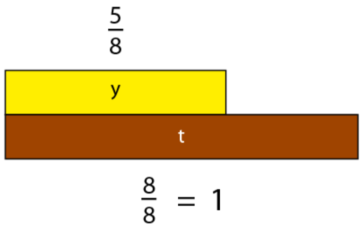
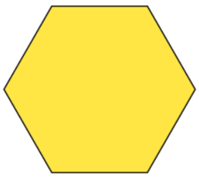
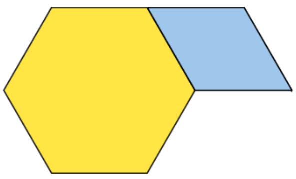
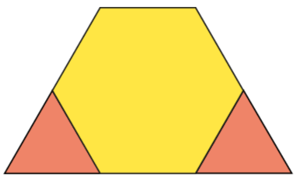
Part	Part as a fraction of the whole	Number of equal parts in the whole	Whole
	$\frac{1}{5}$	5	

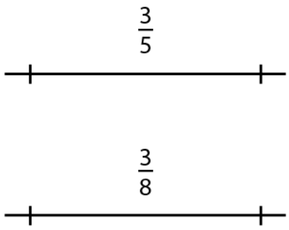
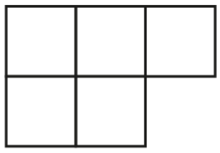
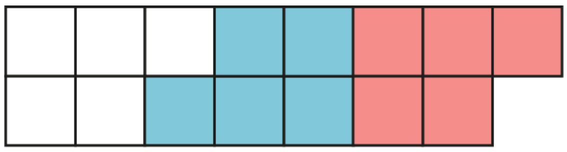
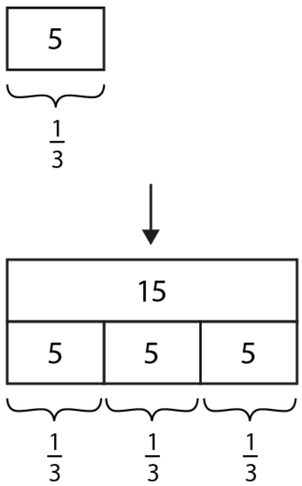
Part	Part as a fraction of the whole	Number of equal parts in the whole	Whole
	$\frac{1}{6}$		
	$\frac{1}{7}$		
		8	
		10	

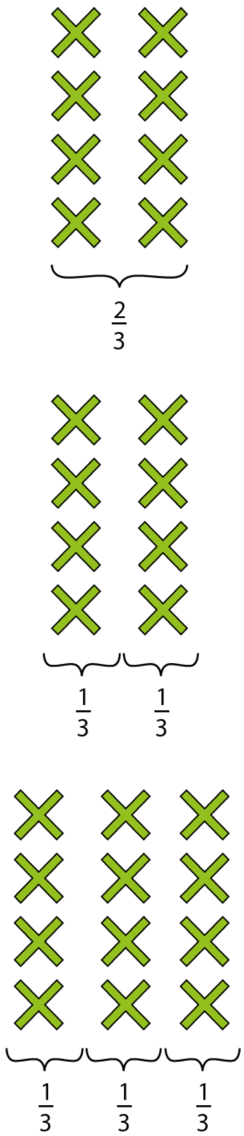
5:2	Now use all three models (linear, set or quantity and area) to provide problems that deepen and consolidate prior learning.	Linear model: *'Which has the larger whole?'*  Quantity model: *'This is $\frac{1}{4}$ of Daisy's sweets. How many sweets does Daisy have?'*  Area model: *'Here is $\frac{1}{3}$ of a shape. Draw what the whole shape might look like.'* 
5:3	Now progress to moving from a non-unit fraction to finding the whole. Children need to understand this as a two-step process: • Step 1: moving from a non-unit fraction back to a unit fraction • Step 2: moving from a unit fraction to the whole (revised above). Now focus on *Step 1*. Provide children with a question that uses an area model, such as the example opposite. Discuss with children that $\frac{3}{5}$ of the shape is already represented, that is, three of the five equal parts. Therefore, we need to draw one of these parts to move from a non-unit fraction to a unit fraction.	*'This is $\frac{3}{5}$ of a shape. Draw $\frac{1}{5}$.'* 

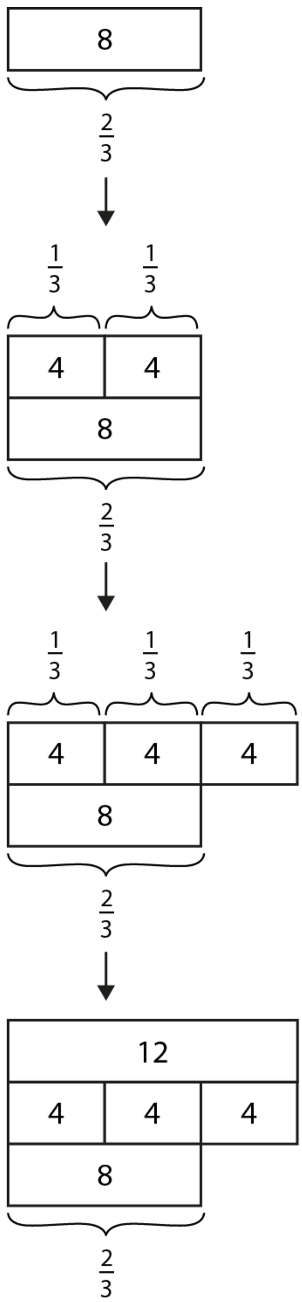
5:4	Provide further similar problems. Vary the examples by: • differing the shape of the unit fraction • using a quantity model • using a linear model • using real-life contexts. It is important to make reference to the relationship between the non-unit fraction and the unit fraction, for example: • *'Five stars represent* $\frac{5}{8}$*.'* • *'*$\frac{1}{8}$ *must be one star.'* Note that at this stage we do not want to draw attention to fractions as operators. Instead, encourage children to identify, for example, eight equal parts, and select one of them.	• *'This is* $\frac{4}{5}$ *of a shape. Draw* $\frac{1}{5}$*.'*  • *'This is* $\frac{5}{8}$ *of a shape. Draw* $\frac{1}{8}$*.'*  • *'Five stars represent* $\frac{5}{8}$*.'* • *'*$\frac{1}{8}$ *must be one star.'* • *'This is* $\frac{2}{3}$ *of a ribbon. Draw* $\frac{1}{3}$ *of the ribbon.'* 
5:5	Progress to fractions of sets where one part contains more than one item, such as the example opposite. Notice how this is slightly more challenging because children need to understand that each third contains four crosses. If children struggle here, reinforce the concepts explored in segments *3.1 Preparing for fractions: the part–whole relationship* and *3.2 Unit fractions: identifying, representing and comparing*, explaining that the representation does not show the entire whole. There are two equal parts shown, out of three equal parts altogether. Therefore, each equal part contains four crosses.	*'This is* $\frac{2}{3}$ *of a set. Draw* $\frac{1}{3}$*.'* 

5:6	Now combine step one and step two. Return to the representations from step *5:3*, where children found the unit fraction from the non-unit fraction. This time, ask children to find the whole. Start with a bar model, such as the one opposite that shows $\frac{3}{5}$. Explain to the children that they found what $\frac{1}{5}$ looks like earlier, and now they are going to find what the whole might look like. Allow time for children to explore this. Ask them to sketch or model their ideas. In this type of question, area model answers will differ. Explore this through discussion, asking children to compare their representations of the whole. Prompt their discussion with questions such as: • *'Here are three possible answers. What is the same/different about them?* • *'Did anyone draw something different?'* • *'Why doesn't everyone's answer look the same?'*	*'This is $\frac{3}{5}$ of the whole. Draw the whole.'* 
5:7	Now challenge children to find the whole using the representations from steps *5:4* and *5:5*.	• *'This is $\frac{5}{8}$ of the set. Draw the full set.'*  • *'This is $\frac{2}{3}$ of the set. Draw the full set.'*  • *'This is $\frac{2}{3}$ m of ribbon. Draw the whole ribbon.'* 

5:8	Next, explore the concept using real-life problems, including measurements of mass and length. Note: At this stage still do not draw attention to fractions as operators (for example, *'If $\frac{5}{7}$ is 15, then $\frac{1}{7}$ must be $15 \div 5$, which is 3, so $\frac{7}{7}$ must be $7 \times 3 = 21$'*). This learning will be explored in segment *3.7 Finding equivalent fractions and simplifying fractions*. Look at the example of Jack's marble collection and encourage children to 'see' the five equal parts in the representation. Ask them how many more equal parts are needed to make the whole and to draw the two missing parts. Cuisenaire® rods can be useful here. For example, show the yellow rod and say *'If this represents $\frac{5}{8}$ of the whole, then show me the whole'*. Use *3.6 Representations*, slide *62* to demonstrate. Similarly, this concept can be further explored using pattern blocks, see opposite. Consolidate and further deepen children's understanding through a range of problems, such as a dòng nǎo jīn.	• *'This is $\frac{5}{6}$ of a metre of rope. Draw the full length of the rope.'*  • *'Jack collects marbles. Below is $\frac{5}{7}$ of his marble collection. Draw the rest of Jack's marble collection to represent the whole collection.'*  • *'If the yellow rod represents $\frac{5}{8}$ of the whole, then show me the whole.'*  • *'The yellow hexagon represents $\frac{6}{8}$ of the whole. Recreate the whole using your pattern blocks.'*  • Possible answers:  

		Dòng nǎo jīn: *'Which has the larger whole? Draw a diagram to prove your answer.'* 
5:9	Progress to working with quantities, to link the set/quantity models children have already seen with bar models. Show the image opposite and ask children to discuss how many squares there would be in the whole shape. Make sure that children write down the written fraction to support their reasoning. Very explicitly make the link between the 'information' contained in the denominator and numerator, and the way they should approach their thinking about this. For example: • *'The whole is divided into three equal parts…'* (Point to the denominator.) • *'…and this is one of those parts.'* (Point to the numerator.) • *'This is one part of three equal parts. So, I need three of these parts.'*	*'If this is $\frac{1}{3}$ of a shape, how many squares will there be in the whole shape?'*  
5:10	Tell the children that you are thinking of a number. Give them a clue and see if they can work out which number it is. The clue is: *'$\frac{1}{3}$ of my number is five. What number am I thinking of?'* Encourage children to make the link between this and the example in the previous step. Show the children how to represent this on a bar model and, as before, make the link to the written fraction:	

	• *'The whole is divided into three equal parts…'* (Point to the denominator.) • *'…and this is one of those parts.'* (Point to the numerator.) • *'We know that five is one part of three equal parts. So, I need three lots of five. That is fifteen.'*	
5:11	Now introduce moving from a non-unit fraction to a whole, via a unit fraction. Return to the image from step *5:5* and reflect on how the children were able to draw the whole set: *'First, we moved from $\frac{2}{3}$ of the set to work out $\frac{1}{3}$ of the set. From this, we were able to work out what the whole set was.'* As before, link this to finding a fraction of a quantity. Present the following scenario: *'I am going to think of a new mystery number. See if you can see the link to what we have just discussed. $\frac{2}{3}$ of my number is eight. What is my mystery number?'* Children should spot that they have, in fact, already worked this out in the visual problem they have just discussed. Again, link this to a bar model and to the written fraction notation. • *'The whole is divided into three equal parts…'* (Point to the denominator.) • *'…and this is two of those parts.'* (Point to the numerator.) • *'We know that eight is two parts of three equal parts. If eight is two parts, then one part must be half of eight. That is four. So, three parts must be twelve.'*	*'This is $\frac{2}{3}$ of a set. Draw the whole set.'* 

	When helping children learn to draw a model to represent a problem like this, they should work towards being able to independently construct models to support their mathematics. To do this, they need to thoroughly understand these models, rather than learn a series of procedures on 'how to draw a bar model'. The children's models may all end up slightly different in appearance. Providing they are 'sense-making' and using the model as a supportive problem-solving tool, then this is acceptable. Finally, show children how to write this chain of reasoning as a series of written statements. • *'$\frac{2}{3}$ of the number is 8.'* • *'$\frac{1}{3}$ of the number is 4.'* • *'$\frac{3}{3}$ of the number is 12.'*	
5:12	Provide plenty of practice moving from a unit fraction to a whole, and especially in moving from a non-unit fraction to a whole. After the children have completed guided practice in class, move to independent practice. Use examples such as those shown opposite, including a variety of questions set in a real-life context and some without a context. The children are highly likely to benefit from continuing to draw bar models	• *'$\frac{1}{3}$ of a number is 15. What is the number?'* • *'$\frac{4}{7}$ of a number is 200. What is the number?'* • *'Zak swam 12 lengths of the pool today. That is $\frac{1}{5}$ of his weekly swim. How many lengths does Zak swim each week?'* (unit fraction to a whole)

	throughout the practice, particularly for moving from non-unit fractions to a whole. If some children feel confident to start working without the model and just write the series of statements, then carefully observe what they do and move them back to drawing a model if they need to.	• *'A gardener has picked apples from 3 of her 8 apple trees. So far, she has 210 apples. Approximately how many apples might she have after picking apples from all 8 trees?'* (non-unit fraction to a whole)