Secondary Magazine 121

National Centre for Excellence in the Teaching of Mathematics

Even though the exam season is in full swing, there's a full half term of the school year remaining. With that in mind this magazine looks at how best to use that time in Year 12 maths lessons, and how other lessons can use learning in the areas of geometry and ratio to develop pupils' ability to reason and construct proofs. As always your views are very welcome, by email to info@ncetm.org.uk or on Twitter, @NCETMsecondary.

Contents

Image credit

Heads Up

Whilst not wanting to define this article as mathematical gossip, it does come close! We've brought together news and current mathematical affairs, all in one place. We do hope it will interest you.

pfeg's My Money Week takes place from 8-15 June. If you haven't taken part before, you might find it well worth looking at. Among material on offer, free, are My Money Week Planners, to help you plan a whole week's worth of activities using pfeg's resources. There's also a competition, for pupils to create a comic strip around the subject of digital finance.

The process, being conducted by the exams regulator Ofqual, of checking the levels of difficulty of sample questions developed by exam boards in preparation for the new maths GCSE (first teaching: September 2015; first exams: Summer 2017) has inched forward a little. A statement from Ofqual at the end of April said its research exercise, announced January, was complete and that it was now considering what action to take as a result. A decision was promised in 'mid-May.' So it should appear on the Ofqual website any day now.

Recently arrived on Twitter is a weekly chat offering help with teaching or revision ideas on any maths topic. It's on Sunday evenings between 7 and 8 pm under the hashtag #mathsTLP and hosted by @mathsjem and @solvemymaths. The chat is then collated into a blog hosted by solvemymaths.com. Hats off to Ed Southall and Jo Morgan for this creative initiative.

Have you seen the latest update from the Core Maths Support Programme, which is now about a year old? There's plenty of useful information here, particularly if you or your school are thinking of participating in this developing area of maths for those having left GCSEs behind.

Did you know that the London Mathematical Society (LMS) provides opportunities for schools/teachers to bid for grants of up to £400 to support teachers with maths specific CPD? There are certain conditions that need to be met and application deadlines for grants are 31 August, 30 November, 31 January and 30 April each year. These grants are available for all teachers.

The latest round of Royal Society Partnership Grants aimed at helping schools run projects in partnership with scientists, engineers or mathematicians, is now open. The grants, for any project with a STEM flavour, are of up to £3,000.

Image credit

Building Bridges

Like a bridge over troubled water, this article explores the transition from parrot-recall and repetition of known facts within geometry, to the complexities of demonstration and mathematical proof – a journey that leads to deeper conceptual understanding and, one hopes, engenders a wanderlust for further travel.

Ask Year 7 pupils "How many degrees do the angles in a triangle add up to?" and most will happily regurgitate "180°".

Ask the question "Why?" and you get a different response ... usually a blank look up at the ceiling … nothing registers ... tumbleweeds roll ...

Ask "how many degrees do the angles on a straight line add up to?" and the answer will undoubtedly be "180°". Your next question would then be "Why?" … and you can guess what happens next! Quite often the response is "because Miss told me", or "because it just is". Now ask "what is a degree?" and that is a real conundrum!

Think of what similar questions could be: "How many degrees do the angles around a point add up to?" "How many degrees do the angles in a quadrilateral add up to?" Knowledge of the angle properties of straight and parallel lines, knowing that angles around a point add up to 360° (which is four right angles), recalling accurately and reliably angle facts about triangles and polygons: all these underpin the higher level skills needed to understand and apply complex ideas such as the "angles in circles" theorems. The earlier we spark pupils' curiosity, the more interest and enthusiasm they develop - and the more accessible the later, harder reasoning becomes.

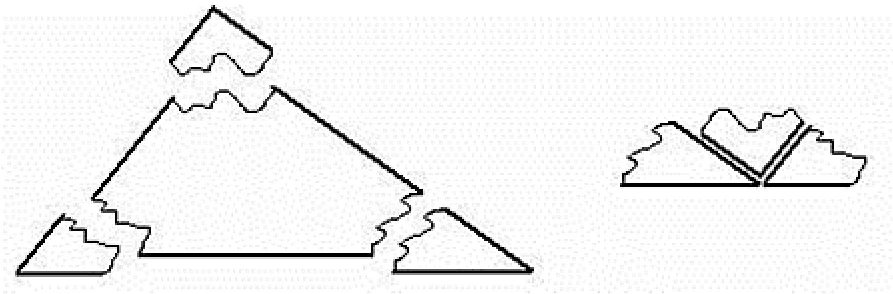
Remember, recall and regurgitate: these are the thinking skills that relate to the lower end of Bloom's Taxonomy. Pupils can remember and recall what they have been told is true, but usually few ask "why?" by themselves: precise questioning by teachers is key to building bridges.

Take a look at the Mathematical Association Postcard Sets for higher and foundation level; they are especially useful at this time of year for revision and memory jogging.

The Demonstration

So how do we help pupils develop their deeper understanding? The use of computer demonstrations is an obvious next step, and the excellent range of animated gifs available to support understanding is well worth exploring: suggestions are at the end of this article. Physical demonstrations are helpful too. You could link this to triangle constructions first to start to build your bridges: check out Mr Collins Comic Book Constructions.

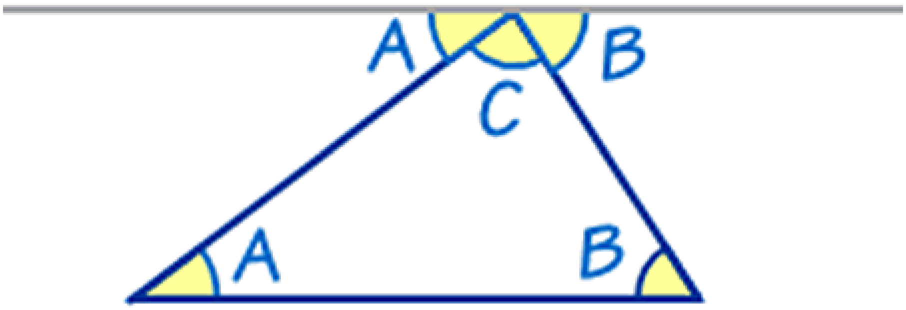
If you tear off the corners of your triangles and place them together, they seem to form a straight line: ta da...



But do they really? And if they do, why do the angles on a straight line add up to 180°? And so the sequence of questioning continues – a good cue to remind ourselves of Malcolm Swan's excellent eight effective principles of maths teaching of which "use of questioning" is but one.

The Proof

We need to use the angle properties of parallel lines. "But why are these true?" – well, that's your cue to talk about Euclid's axioms and the parallel postulate (the 5th axiom), and all the unsuccessful attempts of mathematicians to prove it from the first four axioms. And then, let's consider:



By definition angle A + angle B + angle C = the sum of the angles in the triangle. Angles A are equal and angles B are equal as they are alternate angles defined by a pair of parallel lines. Therefore the sum of the angles within a triangle will equal the sum of the angles on a straight line; but we still need to explain why both equal 180°!

So start your lesson with that simple-seeming question "Why do the angles in a triangle add up to 180°?" and see where it takes you … hopefully, like a bridge over troubled waters (apologies Simon and Garfunkel) the ensuing reasoning will ease (and enlighten) your pupils' minds.

Helpful Resources

- Angles In 2D Shapes from Great Maths Teaching Ideas
- Interior Angle Sums of Triangles, using Geogebra
- NRICH triangle resources
- Spot the Angle! A Mathspad activity showing that with limited information you can still deduce a lot
- Mr Reddy's Geometry Toolbox, a simple interactive tool for constructions.

You can find previous *Building Bridges* features here.

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Sixth Sense

Sixth Sense 121

With past papers complete, exams underway, and students fully focussed on the here and now, this seems like a good time to ponder: "what shall we do with Year 12 when they come back after the modules?"

It has always struck me as unsatisfactory that during Year 12 students don't make much progress – certainly not to halfway – on the overall A-level "calculus journey", especially if they followed any GCSE or extension course that introduced differentiation in Year 11. They may well arrive in the sixth form knowing a rule for differentiating $f(x) = kx^n$; hopefully (especially if they're sitting C2!) they now know how to reverse this process, and they know that doing so helps them answer two seemingly-unconnected questions: "what did I differentiate to get this function I'm integrating?", and "what's the area enclosed between this curve and the *x*-axis?"

Take a deep breath and consider how much there is for your students to learn, understand and practise between now and next year's A2 exams (depending on board, and whether or not they're doing any Further Mathematics): the chain rule, the product rule, the quotient rule, calculus involving exponential and logarithmic function, calculus involving trigonometric functions, implicit differentiation, integration by substitution and by parts, solving differential equations, parametric differentiation, everything to do with hyperbolic functions, reduction formulae … And breathe out ...

So, it seems to make sense to try and make some headway between now and the end of term, and I will start by tackling the chain rule. However, I'm convinced that they're not familiar enough with functional notation to cope with

$$\left(fg(x)\right)' = f'\left(g(x)\right) \times g'(x)$$

nor fluent enough with the operator $\frac{d}{dx}$ to justify telling them that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Rather than starting by asserting these arcane statements, let's approach the problem head on, and find the formula for the gradient of the tangent for each of the following:

1 $y = (2x-3)^3$

(by expanding, differentiating and factorising; students may need nudging in this direction)

2 $y = \left(x^2+2\right)^3$

(again, by first expanding).

Students can then claim and verify (using the same "expand, differentiate, factorise" technique) a prediction for the gradient formula for

3 $y = \left(4x+5\right)^3$

and

4 $y=\left(3x^3+1\right)^4$

Having tried a few of these, students can believe with some confidence:

Prior knowledge: $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

New knowledge: $\frac{d}{dx}\left(\left(function\right)^n\right)=n\left(function\right)^{n-1}\times$ *function differentiated*

In my scheme of work I'll now allow time for lots of practice: finding equation of tangents and normals, location and nature of stationary points and points of inflection, and, in particular, questions such as

What did I differentiate to get $16x\left(x^2-5\right)^3$?

Students will be encouraged to "guess and check": I won't say that a suggestion is "wrong", instead my response will always be "fine, differentiate it and see what happens". I'm aiming for this thought process (if not this layout):

Guess: $\left(x^2-5\right)^4$

Differentiate: $4\left(x^2-5\right)^3\times 2x$

Thought: We need to multiply this by 2 to get the correct answer

Therefore: $\int 16x\left(x^2-5\right)^3 dx=2\left(x^2-5\right)^4+c$

There's lots of practice needed here.

While the new "power of a function" rule beds in, but continuing the theme of gradient rules, next I will introduce the function e^x. I like doing this as an investigation, and there's plenty of scope for IT use here too.

I anticipate that the conversation will go something like this:

Me: First, let's sketch $y=2^x$. What's the gradient of the tangent to the curve $y=2^x$ at the point where $x=0$?

Student A: Differentiate to get $x\times 2^{x-1}$ and put in 0.

Student B: No, because then the gradient would be 0 and it isn't: look at my sketch.

Student A: Does our rule only work if the power is a number, not an *x*?

Me: Let's draw a picture [using Geogebra, desmos, or similar], draw a tangent and estimate the gradient that way.

Student C: Ok, I get 0.7.

Student D: Yes, so do I.

Me:	Why are we so trusting of the software? How else could we estimate the gradient of the tangent?
Student D:	We could draw some chords and work out their gradients.
Me:	Let's do so.
Student B:	Can we use a spreadsheet?
Me:	Good plan.
Student B:	Done it. I'm getting 0.7 too.
Me:	Ok. What's the gradient of the tangent to the curve $y = 3^x$ at the point where $x =$ 0?
Student E:	Computer says 1.1.
Student B:	Excel does too.
Me:	So what can we deduce from these two results?

Students will discuss, ponder and – fingers crossed – suggest the following conjecture (though this may require some steering!):

Conjecture: It's reasonable to assume that there's a number between 2 and 3 which has the property "the gradient of the tangent to the curve $y =$ (that number)^x at $x = 0$ is equal to 1"

Me:	Ok, let's hunt down that number!

And then, in a flurry of graphing software and / or Excel-exploring, the first few digits of e are discovered.

Me:	Right, now let's find the gradient of the tangent to this new-found curve $y = 2.718^x$ at $x = 1,\ x = 2$, etc. What do you notice?
Students:	The gradient of the tangent is the same as the *y*-value.
Me:	In other words …
Students:	Ah ha! $\frac{d}{dx}\left(e^x\right) = e^x$

(I've possibly moved into a world where students say exactly what I hope they will say, so will stop there.)

Next, we will investigate (supported by IT) gradient rules for

(a) $y = 4e^x$ (b) $y = e^{5x}$ (c) $y = e^{x^2}$

where claims will be made and verified, revising in (a) transformations of graphs and in (b) first rewriting as $y = \left(e^x\right)^5$ and then using the recently-learned rule for $y = \left(function\right)^n$.

(c) is starting to generalise this rule, from "power of a function" to "function of a function", and it seems to be saying that

$$\frac{d}{dx}\left(e^{function}\right) = e^{function} \times \text{ function differentiated}$$

I will support this by revisiting the earlier "power of a function" rule, and helping the students to think of $(3x-1)^3$ as "$cube(3x-1)$", where the function $cube(x)$ differentiates to $3 \times square(x)$. Then we will read our "power of a function" rule as a "function of a function" rule – and one that is consistent with the claim just made.

You can imagine similar explorative lessons that look at $\ln x$ and $\ln(function)$, $\cos x$ and $\cos(function)$, $\sin x$ and $\sin(function)$, etc. Tempting as it will be to present a shortcut to a formal statement of the chain rule, developing numerous "function of a function" rules over time makes it far more likely that the students will develop both conceptual understanding and procedural fluency, and they will conclude for themselves that

$$\textit{If we know that } \frac{d}{dx}\left(f(x)\right) = f'(x) \textit{ then it follows that}$$
$$\frac{d}{dx}\left(f\left(function\right)\right) = f'\left(function\right) \times \textit{ function differentiated}$$

When that comes from them rather than from me, I know that the route we've taken along this stretch of the calculus journey, though not the shortest, has certainly been the right one.

John Partridge is Assistant Head at Kings College London Maths School. He runs much of the school's outreach and CPD activities, including a programme for teachers new to teaching further Pure modules. Further information can be found here.

You can find previous *Sixth Sense* features here.

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Secondary Magazine 121

National Centre for Excellence in the Teaching of Mathematics

From the Library
Teaching Calculus with Technology

How do you introduce and teach calculus?

Do you start fairly informally exploring rates of change and gradients of tangents/curves and then move on to the rules of differentiation, then integration both as the inverse operator and as a way of calculating areas under curves, while always focussing on knitting together all the required prior knowledge from algebra, functions and geometry with the new material?

Or do you extend this informal approach to a more formal approach to differentiation linking the tangent as a limiting secant (geometrical representation), the gradient of the tangent as the limit of a sequence of gradients of secants (numerical representation) and the derivative as the limit of a difference quotient (algebraic representation)?

Do you also develop a more formal approach to integration linking the area under a curve to the limiting sum of areas of rectangles (geometric representation), the limit of a series of products (numerical representation) and the integral as the limit of a sum of products (algebraic representation)?

How much do you state, how much do you demonstrate and how much do you prove, including the Fundamental Theorem of Calculus?

Do you go as far as the formal 'epsilon-delta' type analysis normally first encountered in undergraduate Mathematics courses in the UK?

Whichever approach you take and however you develop it, the likelihood is that you will support your approach with the use of technology both to improve and increase the speed of understanding and to enhance visually the links between the geometric, numerical and algebraic representations that pupilss need to make if they are to develop conceptual understanding as well as procedural fluency.

In Teaching and Learning Calculus with Free Dynamic Mathematics Software Geogebra by M Hohenwarter, J Hohenwarter, Y Kreis and Z Lavicka and presented at ICME 11 in 2008, the creator of Geogebra, now based at the University of Linz in Austria, and his colleagues consider in depth six different approaches to the use of Geogebra in teaching calculus, as well as reviewing the teacher-centred and pupil-centred approaches to using technology in the classroom. The paper emphasises the collaborative potential of the use of free, open-source resources such as Geogebra as well as the more obvious financial benefits.

Further reading on approaches to calculus teaching to develop conceptual understanding rather than simple reliance on learning a set of procedural rules, and an extensive bibliography, can be found in Key Ideas in Teaching Mathematics: Research Based Guidance for Ages 9-19 (Anne Watson, Keith Jones and Dave Pratt) 2013 OUP, in particular, Chapter 9 Moving To Mathematics Beyond Age 16. Indeed, the whole book is recommended reading for all mathematics teachers in its entirety.

As a final thought, it is worth noting that that the proposed content for the revised A Level Mathematics from September 2017 specifically includes both differentiation and integration from "first principles". This will possibly require a more formal approach than that sometimes currently adopted in AS Level and A

Level Mathematics teaching, but one that will be equally well supported by technology as discussed in the research paper above.

Footnote: in earlier issues of the Secondary Magazine, we featured a set of six articles by the authors of *Key Ideas in Teaching Mathematics*, following the themes of each chapter. They appear in Issues 105, 106, 107, 108, 109 and 110.

You can find previous *From the Library* features here.

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It Stands to Reason

Hone your pupils' reasoning skills about ratio and proportion

Understanding ratio is one of the nine basic maths skills that can cause major headaches in the classroom.

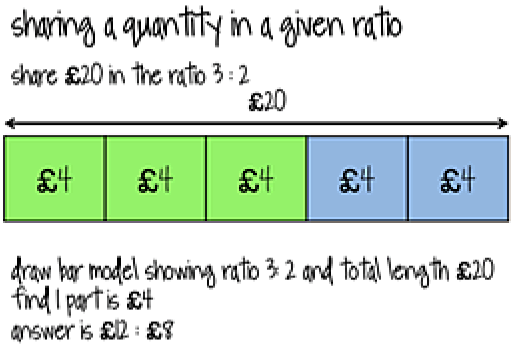
In the New GCSE Curriculum, “Ratio and Proportion” has now been defined as a skill area all of its own within the GCSE subject content and assessment objectives, such is its importance. As in all skill areas, pupils must:

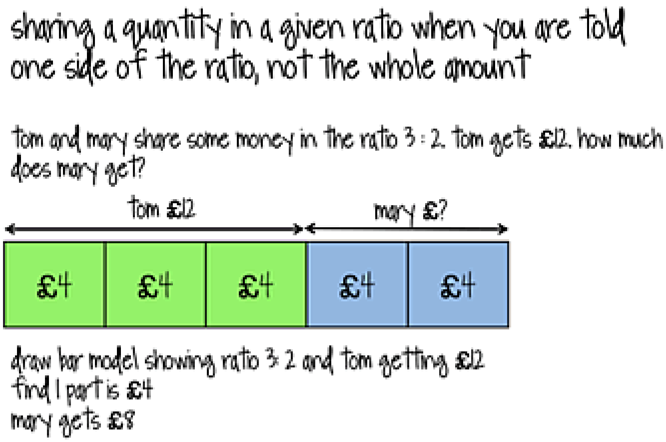
1. develop fluent knowledge, skills and understanding of mathematical methods and concepts
2. acquire, select and apply mathematical techniques to solve problems
3. reason mathematically, make deductions and inferences and draw conclusions
4. comprehend, interpret and communicate mathematical information in a variety of forms appropriate to the information and content.

The 16 objectives within the new curriculum clearly define the links between ratio and many other areas of mathematics, highlighting the interconnected nature of the subject matter: for example, links are made at the higher level with trigonometric ratios and compound interest.

Bar Modelling

William Emeny (@Maths_Master) has written a superb blog at Great Maths Teaching Ideas on the use of bar modelling and its application to ratio and proportion problems. His presentation is beautifully clear and legible: handwriting to die for! (go to the blog for a larger version of the examples below).



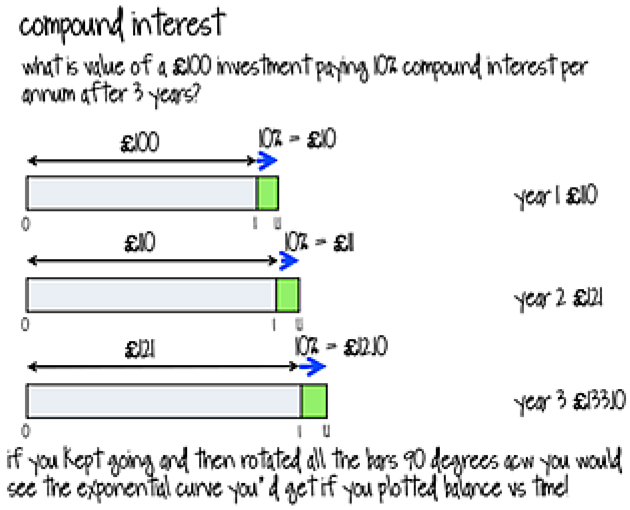


The blog begins with an embedded video that is well worth watching. The content is for primary pupils, but the excellent inquiry-based pedagogy that is modelled by the presenter is applicable right through to KS5, and beyond.

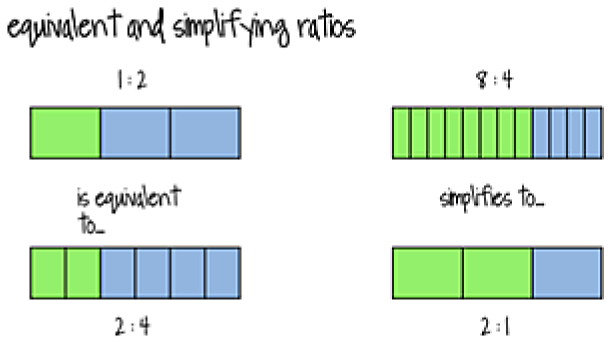
Having used several of the approaches I've drawn together in this article with pupils and with trainee teachers, I can see the huge benefit there is to having these tools to use. In the case of the bar model, it is a tool that gives consistency of approach to many maths problems, from easy to complex levels.

Bar Modelling, Algebra Tiles and Arrow Diagrams are not in themselves a panacea to solving all problems, including world peace, they are certainly very helpful concrete visualisations that give pupils routes into and out of problems that are usually found to be difficult.

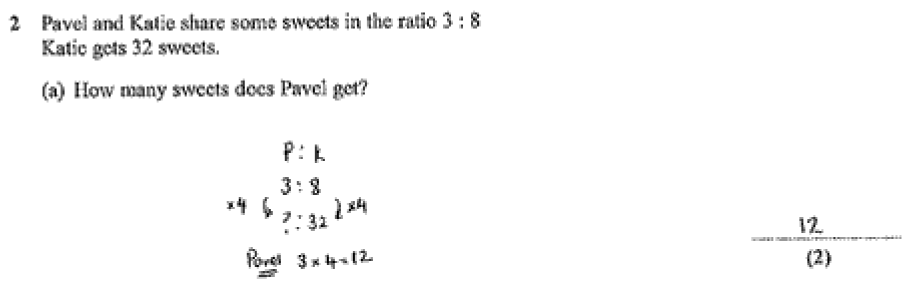
Here you can see bar model solutions to a higher level compound interest question as well as a lower level equivalency and simplifying ratios. Both are beautifully demonstrated and drawn by @Maths_Master himself:



It works for ratio too:



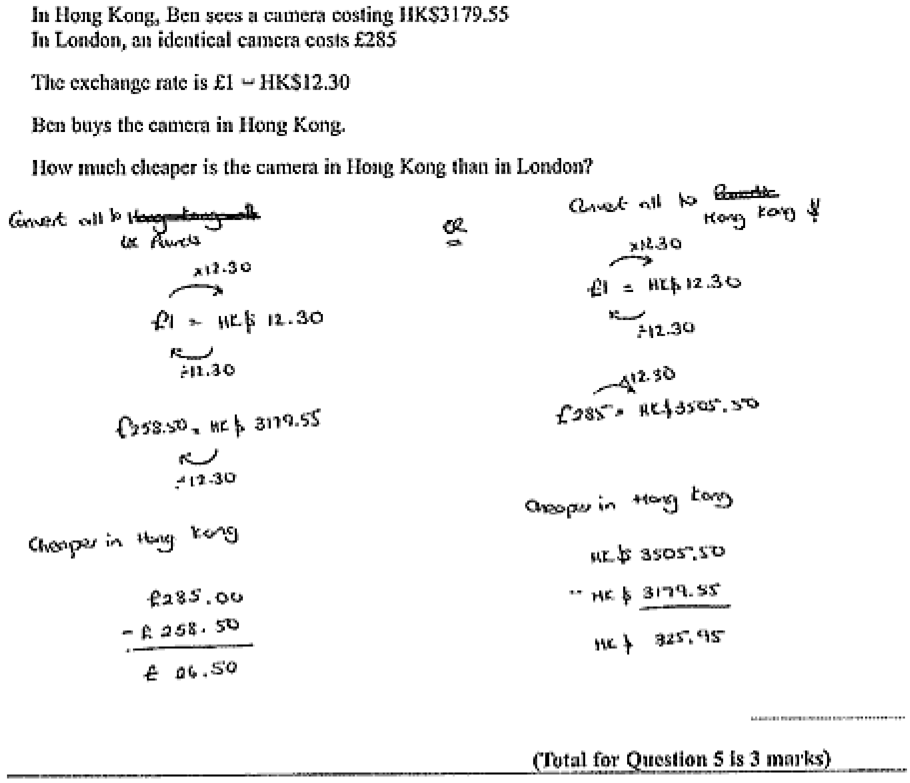
Arrow Diagrams


2 Pavel and Katie share some sweets in the ratio 3 : 8
Katie gets 32 sweets.

(a) How many sweets does Pavel get?

Examples of arrow diagrams can be clearly seen here in exam questions answered by Mr Barton (not quite the legibility of @Maths_Master but still good - again, go to the link for larger versions of the examples). The lower level skill of finding the missing amount given the ratio is aided with the use of the arrow diagram and then recognising the multiplying factor.

In this next example the arrow diagram is used to identify the multiplying and dividing factors so that the pupil can then apply them to solve an unstructured problem:


In Hong Kong, Ben sees a camera costing HK$3179.55
In London, an identical camera costs £285

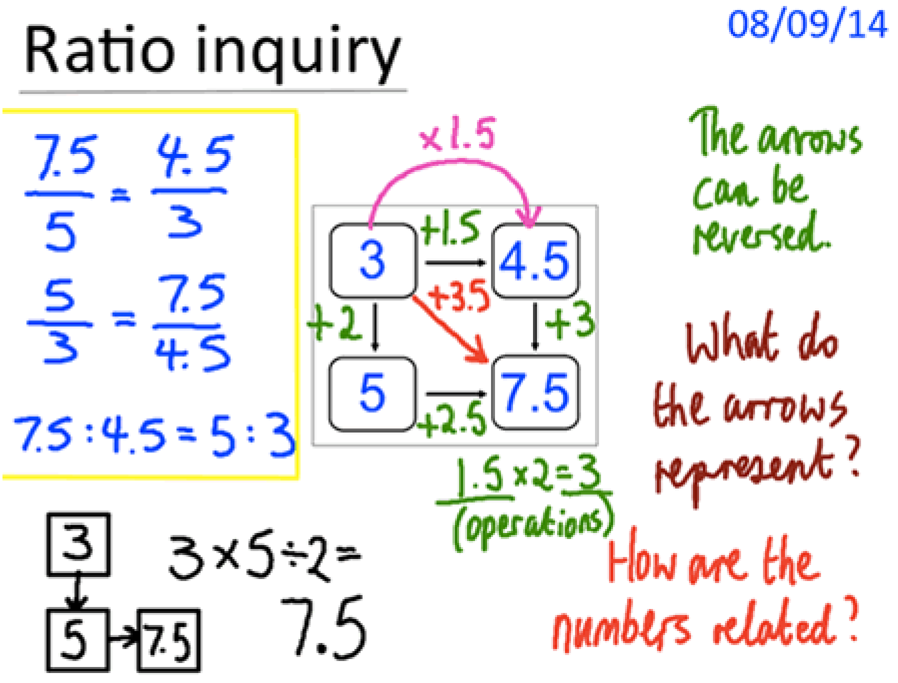
The exchange rate is £1 = HK$12.30

Ben buys the camera in Hong Kong.

How much cheaper is the camera in Hong Kong than in London?

(Total for Question 5 is 3 marks)

Here - from Inquiry Maths - you can see the Arrow Diagram in action as pupils react to the Inquiry Prompt Question "What do the arrows represent?"



Resource recommendations

Durham Maths Mysteries
The Ratio and Proportion Mystery is an old favourite but an excellent one, with 11 clues covering ratio, fractions, percentages and average. The mysteries cover a lot of maths content, and they deepen pupils' understanding as they use the clues to work backwards to find the answers.

Nuffield Free Standing Maths Ratio Bingo & Matching Cards
Practise the skills required within a good game of bingo: all the threes, two little ducks, clickety click...

NRICH and ratio
101 things to do with ratio & proportion

Just Maths
A challenging functional-style question.

Read previous *It Stands to Reason* features here.

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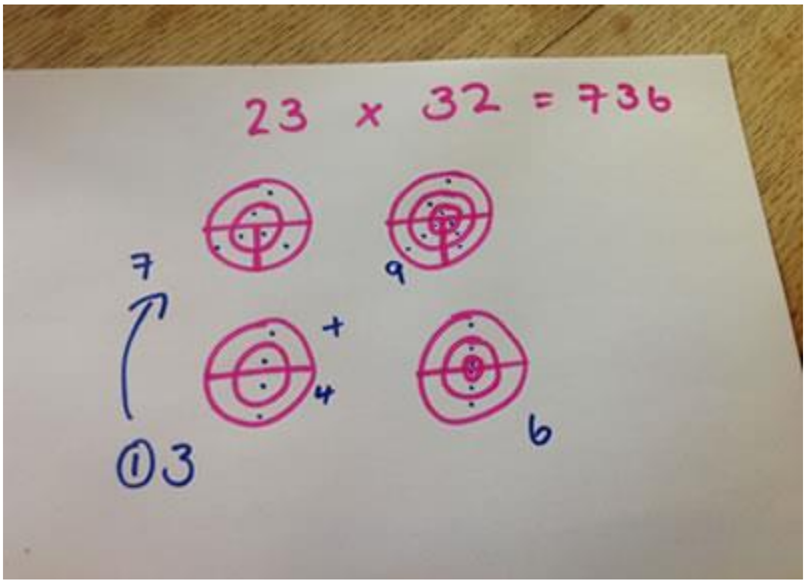
Eyes Down

Eyes down...your Starter for Ten...it's the monthly picture that you could use with your pupils, or your department, or just by yourself, to make you think about something in a different way.

Vedic Multiplication - a celebration of diversity within the maths classroom.
Here is a picture of a form of Vedic multiplication. The answer is 736 but how does it work?



Ask your pupils if they can figure it out.
Ask if there are any other ways they could calculate 23 x 32.
What about the Egyptian method...the Russian Peasant Algorithm...or Napier's Bones?
Gypsy Hand Multiplication is another favourite which helps ease time tables.

Are there any other radically diverse methods for calculations your pupils can discover?

Looking at the different forms (or even just one, such as Vedic multiplication), you could challenge your pupils to multiply two 3-digit numbers, or two non-integers.

There are at least 16 different ways of tackling multiplication! The pupils in your class may have a variety of methods between them and this is an opportunity to celebrate difference - both difference in method and difference in culture - at the same time as deepening their conceptual understanding. Do you have any alternative methods you wish to share that may prompt discussion? Let us know info@ncetm.org.uk.

If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb) to us at info@ncetm.org.uk, with a note of where and when it was taken, and any comments on it you may have.

Read previous *Eyes Down* features here

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