Secondary Magazine 126

National Centre for Excellence in the Teaching of Mathematics

Welcome to Issue 126 of the Secondary and FE Magazine

Falling leaves and sky-rocketing pumpkin prices mean that the end of October is fast approaching: we hope that the first half of the first term has gone smoothly and successfully, and that both you and your pupils are fully into your strides – especially so if you're in a new post or a new school, and even more especially so if you're an NQT. If you're taking stock over the break and have some "what I wish I'd known in September" advice you want to share with us and your peers, please get in touch - email info@ncetm.org.uk, or @NCETMsecondary on Twitter. Do also get in touch if you want to share a picture for the regular "Eyes Down" article: if your picture is published, we'll send you a £20 voucher.

Contents

Image credit
Page header by Tanya Hart (adapted), some rights reserved

Heads Up

Whilst not wanting to define this article as mathematical gossip, it does come close! We've brought together news and current mathematical affairs, all in one place. We do hope it will interest you.

The schools hosting the Shanghai exchange teachers are publishing their programmes of events and PD opportunities in November: do contact your local host school, or your Maths Hub Lead school, to find out what's on offer. Some "first impressions" from the NCETM team who visited Shanghai schools in September, and a flavour of how one school plans to work with their visiting teachers, have been posted on the Maths Hubs website.

If you're looking for ways of assessing diagnostically the depth, security and confidence of your pupils' understanding of key mathematical concepts, we're sure you'll find the NCETM's recently published assessment materials useful. The Director for Secondary explains here why he feels they are relevant to KS3 teachers.

There's lots of discussion – sometimes little more than water-cooler chit-chat – about applying research findings from cognitive science in the classroom. This blog pulls together evidence not hearsay: well worth reading.

Many of the Maths Hubs are running programmes to develop Professional Development Leads. If you'd like to find out more about this training, and how it could be beneficial for you and your career, there is more information here.

Following last month's article about schemes of work for GCSE post-16 resit students, we've had our attention drawn to Citizen Maths. This is an online self-study programme for people in work who want to improve their confidence in and grasp of maths at level 2, but it could also help post-16 GCSE students as they prepare for their resit exams.

Five new dates for maths workshops run by the NCETM, aimed at teachers and trainers in the FE and Skills sector, have been announced. They are all in December: details are under the *Deepen Your Understanding of Number* heading on our FE Workshops page.

There's been a lot of media discussion – not all equally well-informed, to put it mildly – about teacher recruitment. We're keen to hear your thoughts on this topic: is your maths department able to recruit successfully? If not, have you changed your recruitment tactics, and if so has this been effective? Are you considering, or already trialling, different models for allocating maths teachers and teaching assistants to classes? Let us know: is teacher recruitment a crisis or an opportunity? You can email info@ncetm.org.uk, or share on Twitter, @NCETMsecondary.

Image credit
Page header by NS Newsflash (adapted), some rights reserved

Building Bridges

With Hallowe'en *memento mori* all around us, we've been shuffling along this mortal coil and reflecting on our own mortality; well, about Actuarial Life Tables to be precise. These data sets might sound dull and lifeless (pun fully intended), but they're a rich resource to explore when looking to extend pupils' understanding of probability and risk.

Before exploring an Actuarial Life Table, you should ascertain your pupils' prior conceptual understanding of probability: what are they building on? Almost certainly they will have hitherto written probabilities as the consequence of an actual or imagined experiment: dice will have been rolled, cards will have been dealt, coins will have been flicked. The probability of an outcome will have been defined as the number of times the desired outcome could (or did) occur divided by the total number of possible outcomes that could (or did) occur: for example, either imagining that if we were to flick a fair coin, or arguing that when we actually did flick the fair coin, there would be (or there were) two possible equally likely outcomes of which one, say "Heads", is the one we want(ed), we say that "the probability of getting a Head is ½". If we conduct an experiment investigating the probability of a drawing pin landing on its head (with the point facing upwards) and if when we drop a pin ten times it lands on its head seven times, then we express that this happened by saying that "the (experimental) probability of this drawing pin landing on its head is 0.7". We believe that the more trials are conducted, the more consistent the ratio of "point up: point not up" landing positions will be, and so we expect that a graph of the number of trials on the horizontal axis versus the (experimentally-calculated) probability on the vertical axis will show the probability stabilizing around a consistent value as the experiment is repeated more times. Why do we believe this? What's our justification? These should be discussed with your pupils.

The coin-flicking / dice-rolling train of thought can be extended to look at survival rates in a population: we take a group of people of the same age and see how many are alive at the end of the year, and if, for example, from a group of 100 seventy year-olds, 93 are still alive on the day before their 71st birthday, then we say that the survival rate (for this group) as 0.93 (and the death rate is 0.07). If this group represents a population as a whole – and that is "a big if" that needs careful consideration of gender and both the current and former diet, environment, health, employment, housing etc. of the group's members – we can record this result in a Life Table, the data in which are used by life insurance companies to model the probability of survival to particular ages for a member of the population.

The actuaries who work with the life tables start with the somewhat morbid concept of an imaginary cohort of 100 000 births and then they use the data collected over time from group such as our 100 seventy year-olds to predict – to model – how many will still be alive at the end of each year. For a fuller explanation of life tables, and how to read them, there's this useful guidance from healthknowledge.org.uk.

Your pupils should pause to think about the difference between the actuaries' reasoning and the reasoning from dice-rolling / coin-flicking. Tomorrow's coin is plausibly the same as today's (perhaps it's literally the same coin), and so today's reasoning about flicking the coin will apply plausibly when I flick the coin tomorrow: the reasoning today is projectable into the future. But, in the life table situation, tomorrow's cause of and age at death are not the same as today's (it'll be a different person, for starters), so to what extent is the insurance companies' reasoning from past data projectable? And if the reasoning is projectable, how far is it so? The fair coin of 2050 will probably behave in the same way as the fair coin of 2015 does, but the world of 2050 is likely to be very different to the world of 2015: are predictions about life and death 35 years from now at all plausible?

Your pupils will enjoy exploring a life table such as the one below, which is modelling survival (or, alternatively, death) rates in rural India in 1995-1999. The two highlighted figures show what's called the "life expectancy" of an individual: a striking 60.8 for men and 62.5 for women *(click image to enlarge)*.

India, 1995-99

Age-interval	Total				Male				Female			
x to x+n	$_nq_x$	lx	$_nL_x$	e_x	$_nq_x$	lx	$_nL_x$	e_x	$_nq_x$	lx	$_nL_x$	e_x
					Total							
0-1	0.07464	100000	94455	61.7	0.07458	100000	94458	60.8	0.07478	100000	94574	62.5
1-5	0.02974	92536	363067	65.7	0.02391	92542	364477	64.7	0.03602	92522	361432	66.6
5-10	0.01089	89784	446476	63.7	0.00965	90329	449466	62.3	0.01227	89190	443212	65.0
10-15	0.00608	88806	442682	59.3	0.00578	89457	445993	57.9	0.00643	88095	439059	60.8
15-20	0.00891	88266	439447	54.7	0.00782	88940	443048	53.2	0.01015	87529	435528	56.2
20-25	0.01020	87480	435237	50.2	0.01074	88244	438950	48.6	0.01163	86640	430730	51.7
25-30	0.01317	86587	430171	45.7	0.01317	87296	433696	44.1	0.01311	85632	425387	47.3
30-35	0.01484	85447	424136	41.2	0.01588	86147	427431	39.6	0.01380	84509	419648	42.9
35-40	0.01760	84179	417332	36.8	0.02035	84779	419778	35.2	0.01455	83343	413761	38.5
40-45	0.02402	82698	408775	32.4	0.02836	83054	409691	30.9	0.01937	82130	406861	34.0
45-50	0.03376	80711	397164	28.2	0.04022	80698	395859	26.7	0.02647	80539	397711	29.6
50-55	0.05319	77987	380200	24.0	0.06194	77452	375964	22.7	0.04379	78408	384037	25.3
55-60	0.07705	73839	355806	20.3	0.08959	72655	347904	19.1	0.06417	74974	363623	21.4
60-65	0.12208	68149	320981	16.7	0.13971	66145	308649	15.7	0.10539	70164	333393	17.7
65-70	0.17261	59829	274362	13.7	0.19456	56905	257771	12.8	0.15252	62769	291072	14.4
70+	.....	49502	544126	11.0	.....	45833	471531	10.3	.....	53196	614672	11.6

Life tables for different countries can be downloaded from lifetable.de. Your pupils can analyse the data that the tables are modelling and consider what's the same and what's different between different countries. What do they observe when they look at the data by gender (women's life expectancy is four years greater than men's in England, for example), or by age range (which periods are the riskiest in a person's life, and why are these so?) or over time (how have the life table data for one country changed over, say, the last 20 years?)? *(click image to enlarge)*

Interim Life Tables, England — Back to contents

Period expectation of life
Based on data for the years 2006-2008 — Office for National Statistics

Age	Males					Females				
x	m_x	q_x	l_x	d_x	e_x	m_x	q_x	l_x	d_x	e_x
0	0.005393	0.005379	100000.0	537.9	77.74	0.004436	0.004426	100000.0	442.6	81.88
1	0.000402	0.000402	99462.1	40.0	77.16	0.000347	0.000347	99557.4	34.6	81.24
2	0.000237	0.000237	99422.1	23.6	76.19	0.000204	0.000204	99522.9	20.3	80.27
3	0.000165	0.000165	99398.6	16.4	75.20	0.000163	0.000163	99502.6	16.2	79.28
4	0.000138	0.000138	99382.2	13.7	74.22	0.000115	0.000115	99486.4	11.4	78.30
5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

If your pupils have previously considered joint probabilities for combined events, they could use the life table to model probabilities such as that of a married couple both reaching a certain age. This would be an interesting context in which to consider the independence of the events: do your pupils think that assortative mating is likely to increase or decrease the probability of a married couple both reaching a certain age? Let us know (info@ncetm.org.uk or on Twitter @NCETMsecondary) what they discover and how they reason about their observations.

You can find previous *Building Bridges* features here.

Image credits Page header by Alan Levine (adapted), some rights reserved

Sixth Sense

Last month we looked at factorising cubics (and, by extension, higher order polynomials) by adapting the multiplication model of "farmers' fields". I ended by wondering what happens if we try to "factorise" $x^3 - 4x^2 + 2x + 7$ using $(x-3)$ as our "factor"?

As before, we can express the given information as a field and see what happens:

$\times$	$?x^2$	$?x$	$?$	
x	x^3			
-3				
				$x^3 - 4x^2 + 2x + 7$

The colour-coded diagram below shows the following steps:

Step 1 – to get the x^3 term, we just need $1x^2$; this generates the $-3x^2$ term.
Step 2 – since we want a total quadratic term of $-4x^2$, we have to have a $-x$ term in our quadratic "factor"; this also generates the $3x$ term.
Step 3 – since we want a total linear term of +2x, the final entry in the top row must be -1

$\times$	$1x^2$	$-1x$	-1	
x	x^3	$-x^2$	$-x$	
-3	$-3x^2$	$3x$	3	
				$x^3 - 4x^2 + 2x + 7$

Thus our attempt to factorise hasn't worked – we've ended up with a constant term of "3" when a correct factorisation would have given us "7". However, we can instead deduce that:

$$x^3 - 4x^2 + 2x + 7 \equiv (x-3)(x^2 - x - 1) + 4$$

or equivalently,

$$\frac{x^3 - 4x^2 + 2x + 7}{x-3} \equiv x^2 - x - 1 + \frac{4}{x-3}$$

To enable students to use the right language, I often use a numerical analogy, such as "what's 30 divided by 7"? Since

$$30 = 7 \times 4 + 2$$

the answer is sometimes given as "4 remainder 2", or:

$$\frac{30}{7} = 4 + \frac{2}{7} = 4\tfrac{2}{7}$$

Your students will have known since primary school about remainders; they may not know that the "4" in this calculation is called "the quotient". Extending this language to our algebraic example, they can now say that "$x^3 - 4x^2 + 2x + 7$ divided by $x-3$ gives a quotient of $x^2 - x - 1$ and a remainder of 4".

There's an important parallel here between numerical top heavy fractions (if the numerator is greater than or equal to the denominator we can simplify the division, as we did with 30 ÷ 7) and algebraic ones (if the degree of the numerator is greater than or equal to the degree of the denominator we can simplify the division to find the quotient and remainder).

This seems like a good time for some practice: I'd ask students to find quotients and remainders using more fields. Just think how their algebraic fluency will improve as they do so! It's well worth their considering a non-linear divisor: For example, *Find the quotient and remainder when* x^4+7x^3-2x+8 *is divided by* x^2+3x+5, leads to this field (where, as before, each column from left to right gets filled in, in turn, from top to bottom):

×	x^2	$4x$	-17	
x^2	x^4	$4x^3$	$-17x^2$	
$3x$	$3x^3$	$12x^2$	$-51x$	
5	$5x^2$	$20x$	-85	
				x^4+7x^3-2x+8

From which they conclude that

$$x^4+7x^3-2x+8 \equiv \left(x^2+3x+5\right)\left(x^2+4x-17\right)+29x+93$$

and so they say that the quotient is $x^2+4x-17$ and that the remainder is $29x+93$.

Establishing the Remainder Theorem follows naturally. Recall that:

$$x^3-4x^2+2x+7 \equiv (x-3)\left(x^2-x-1\right)+4$$

Since this is an identity, it is true for all values of *x*. In particular, the students can notice that substituting $x=3$ into the original expression (the dividend, on the left hand side) will **have** to give a value of 4, because the right hand side will become 0 × ? + 4 – and they should check that it does.

Having by now seen lots of examples, your students should be confident in making the claim that if they attempt to divide some polynomial $f(x)$ by a linear expression $(x-a)$ they will end up with $f(x)=(x-a)q(x)+r$, where (by experience) the degree of *q* is one less than the degree of *f*, and *r* is constant. Thus they can substitute $x=a$ into both sides to see that $f(a)=r$ by necessity. Do make sure that they learn to quote this result: the importance of "division by a field" is that it gives students conceptual understanding of division, but it's not at all a procedurally fluent or efficient way to find the value of the constant remainder. The Factor Theorem, that $f(a)=0 \Leftrightarrow (x-a)$ is a factor of $f(x)$, is just a special case of the Remainder Theorem; again, students need to learn to quote this, not deduce it "from scratch" every time.

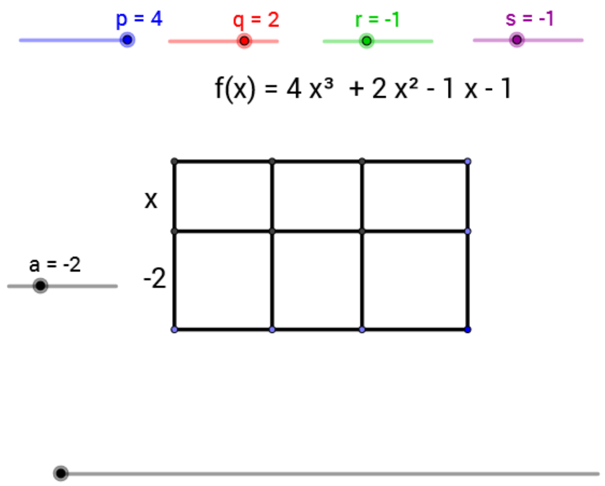
Division by quadratic divisors is usually a Further Maths topic, but it provides a good opportunity for single Maths students to deepen their confidence of the reasoning they've developed, and thus a question such as

> When the polynomial $f(x)$ is divided by $x-3$ the remainder is 8. When $f(x)$ is divided by $x+1$ the remainder is 12. When $f(x)$ is divided by $(x-3)(x+1)$ the quotient is $q(x)$ and the remainder is $rx+s$. Find *r* and *s*.

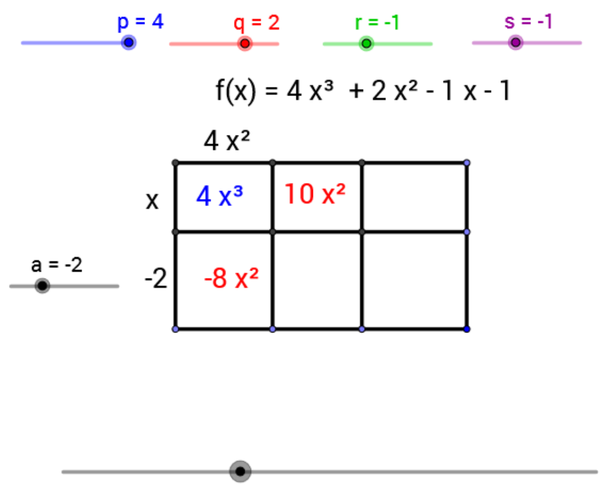
is well worth posing to them. Given all the examples they've considered, I'd expect that my students would be able to generalise from the linear case and produce a solution along the lines of

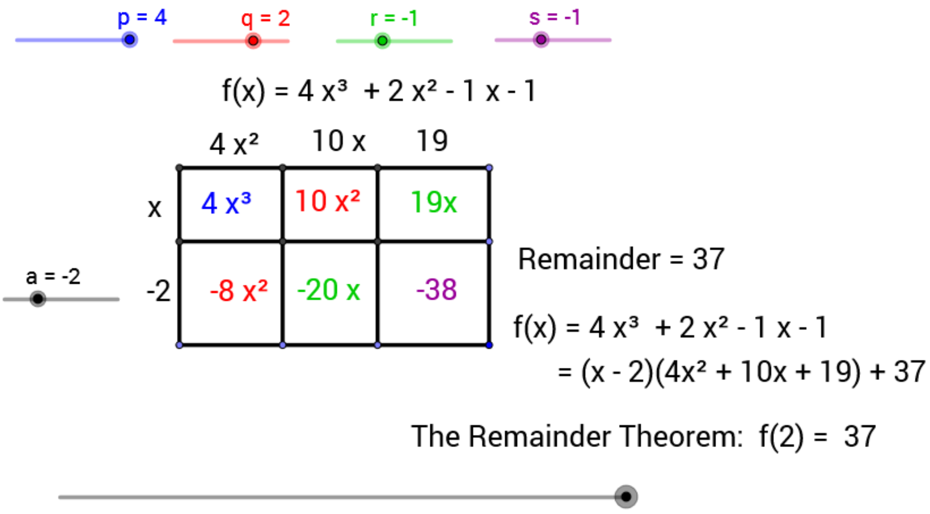
$$f(x) = (x-3)(x+1)q(x) + rx + s$$
$$f(3) = 8 \Rightarrow 3r + s = 8$$
$$f(-1) = 12 \Rightarrow -r + s = 12$$
$$\text{Hence } r = -1, s = 11$$

If you're a frequent user of Geogebra, I'm sure you'll have already envisaged the potential for presenting the "field routine". Bernard Murphy (MEI Programme Leader) got in touch after last month's article to share his polynomial-division file, which allows you to choose your coefficients and your divisor –



- and then runs through the process as you move the slider at the bottom of the screen:





Very neat!

You can find previous *Sixth Sense* features here.

Image credit
Page header by takomabibelot (adapted), some rights reserved

From the Library

In 1994 Eddie Gray and David Tall of the University of Warwick wrote a research paper extolling the virtue of ambiguity in mathematics. They identified that those who can flexibly handle different perceptions of the one mathematical object are those who find mathematics much easier and who seem to achieve at a higher level. Their central ideas revolve around the duality between process and concept. They distinguish "process", as in the "process of addition", and a "procedure", which refers to a specific algorithm for carrying out a process (such as, for addition, "count-all" or "count-on"). And while procedures (routine manipulation of mathematical objects) and concepts (knowledge rich in relationships and connected ideas) are often considered disparate areas of learning, Gray and Tall argue that they are organically connected: procedures are interiorised by the learner and become objects in themselves, in turn manipulated within other procedures.

The authors draw attention to mathematical symbolism, highlighting that the same notation is often used "to represent both a process and the product of that process". Examples they give include:

> The symbol $^{3}/_{4}$ stands for both the process of division and the concept of fraction.
>
> The trigonometric ratio sine A = length of opposite ÷ length of hypotenuse represents both the process for calculating the sine of an angle and its value (see From the Library in Issue 125).
>
> The algebraic symbol $3x + 2$ stands both for the process "add three times *x* and two" and for the product of that process, the expression "$3x + 2$".
>
> The function notation $f(x) = x^2 - 3$ both gives the instructions as to how to calculate the value of the function for a specific value of x as well as encapsulating the complete concept of the function for a general value of x.

The authors argue that "the ambiguity in interpreting symbolism in this flexible way is at the root of successful mathematical thinking"; conversely, the absence of such ambiguity leads to meaningless procedures ("two negatives make a plus", "change sides, change signs"). The reification of the process into a concept, and the ability (which is rooted in the meaning provided by the process) to flexibly shift between the two complementary perspectives, supports the learner to make the crucial step from thinking about the concrete to thinking in the abstract. Gray and Tall name this idea *procept*, "the amalgam of three components: a *process which produces a mathematical object*, and a *symbol* which is used to represent either process or object." Critically, they argue, it is the use of procept that distinguishes performance at mathematics.

Procedural thinking, they explain, focusses "on the procedure and the physical or quasi-physical aids which support it", while proceptual thinking "is characterised by the ability to compress stages in symbol manipulation". Procedural thinking provides guaranteed success in a limited range of situations but is unlikely to lead to success in more complex problems. Gray and Tall identify the proceptual divide, separating those who think procedurally and those who think proceptually. Procedural thinking limits the formation of concepts at the next level up; the proceptual divide creates a chasm between those for whom mathematics provides great power and those for whom it becomes a subject of "spiralling complexity".

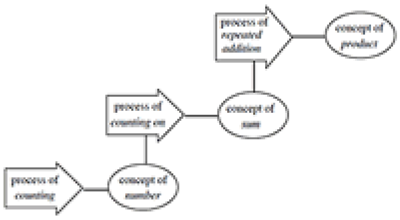


Figure 9 : Higher order encapsulations

Fig 1 from Gray and Tall *(click image to enlarge)*

For example, the authors argue that the procedure of multiplication, e.g. 3 × 4, will be almost impossible for a pupil to grasp while still considering addition as a (separate) procedure, 4 + 4 + 4. In contrast, proceptual thinking collapses the hierarchy of thought into a single level. The symbol 4 + 4 + 4 simultaneously represents the process of adding 4 repeatedly and the sum of 12, enabling progression to the concept of product.

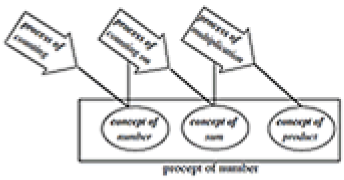


Figure 10: Collapse of hierarchy into operations on numbers

Fig 2 from Gray and Tall *(click image to enlarge)*

Compression of mathematical ideas makes them simpler to handle. The "whole of mathematics may therefore be thought of in terms of the construction of structures … mathematical entities move from one level to another, an operation on such entities becomes in its turn an object of the theory" (Piaget, 1972, p.70). The compression of a process into a compact idea allows the short-term memory to manipulate a mathematical object that is rich in conceptual meaning.

So what does their theory imply in the classroom for teaching and learning? It suggests that new concepts need to be grounded in sufficient examples of the appropriate type to enable the learner to interiorise the procedures, implying, as Weber points out (Weber 2005, p.95), that an opportunity and structure for reflection is required. Fields Medallist William Thurston wrote about the importance of "work[ing] through some process or idea from several approaches" in order to be able to see the mathematical object as a whole. Applying this to the procept of "$^3/_4$", this suggests that pupils need to experience different procedures such as representing as different equivalent fractions and ratios, calculating the fraction as a decimal, as a percentage, as a fraction of something, visualising it as cutting an object (an area, a length, a volume) into 4 equal parts and then choosing 3 of them, experiencing dividing 3 objects between four people. For the function $f(x) = x^2 - 4$, pupils should calculate a table of "input-output" values, sketch the graph of $y = f(x)$ using pencil and paper, solve the equation $f(x) = 0$ and use a dynamic geometry package to slowly trace out the curve. These are just a few suggestions; let us know (info@ncetm.org.uk or Twitter @NCETMsecondary) what you try with your pupils, and what impact and outcomes you observe.

References

Beth, E. W. and Piaget, J. (1966). *Mathematical Epistemology and Psychology* (W. Mays, trans.), Dordrecht: Reidel.

Boaler, J. (2009) *The Elephant in the Classroom: Helping Children Learn and Love Maths*.

Gray, E. M. and Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic. *Journal for Research in Mathematics Education*, 26 2, 115 – 141.

Piaget, J. (1972). *The Principles of Genetic Epistemology* (W Mays, trans.) London: Routledge and Kegan Paul.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics*, Vol. 22, No. 1 (Feb., 1991), pp. 1-36.

Thurston, W. P. (1990) Mathematical Education, *Notices of the American Mathematical Society*, 37 (7), 844 – 840.

Weber, K. (2005) Students' Understanding of Trigonometric Functions, Mathematics Education Research Journal 2005, Vol. 17, No. 3, 91–112

You can find previous *From the Library* features here.

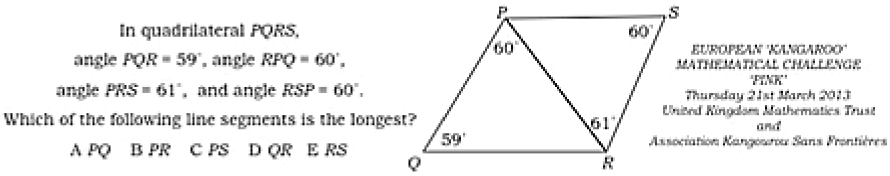
Image credit
Page header by Alexandra*Rae (adapted), some rights reserved

It Stands to Reason

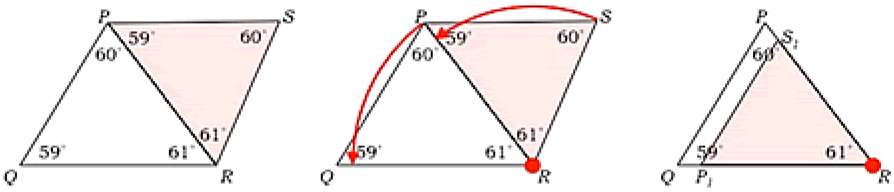
The thoughts about tasks and problems below are a continuation of the discussion in last month's It Stands to Reason about ways of developing pupils' abilities to imagine and recognise rotations. Such abilities help pupils to reason geometrically, which will be tested explicitly in the new GCSE papers.

When, in any mathematical context, pupils work regularly at visualising features, and describing or drawing what they visualise, their powers of imagination (that they already possess naturally) grow. In particular they become more adept when they look at a diagram of a geometrical problem at visualising features that are not yet shown, at perceiving alternative ways of seeing an image, and also at imagining what can be changed without changing essential relationships.

For example, consider the following (rather tricky) problem from the very useful bank of problems provided by the United Kingdom Mathematics Trust (UKMT) *(click image to enlarge)*:



A way of reasoning to the answer becomes clear when the problem-solver visualises the rotation of triangle *PSR* anti-clockwise about *R* through 61˚ to see that triangle *PSR* is similar to triangle *QPR*, and smaller, and remembers that the longest side of a triangle is opposite its largest angle.



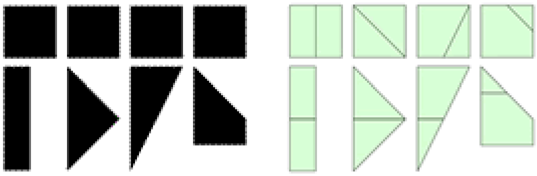
How then can a teacher provide frequent opportunities for pupils to practise visualising (rotations in particular), and describing or drawing what they visualise? A teaching strategy closely related to *Say What You See*, described in last month's issue, is to challenge pupils to create and describe their own examples of something. As pupils begin tentatively to think of, and to describe to their peers, their examples, the goal is to extend the range of the class of objects that the pupils regard as permissible examples, and to stretch their imaginations as far as possible by asking for another example and another one and another and another ….

Another and Another

Very often when you show pupils your own examples, or examples from a textbook, the pupils do not all experience the examples in the same way as you do; that is, they may not focus on what it is that the **author** intended the examples to exemplify. By constructing, explaining to others, and discussing, their **own** examples, pupils can move towards the deep conceptual understanding you are guiding them to acquire.

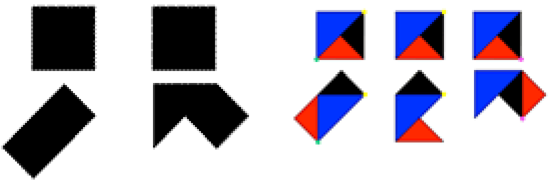
Here are some examples to try.

- Ask your pupils to split a square into two pieces, and then rotate one piece to form a new shape, as in these four examples.

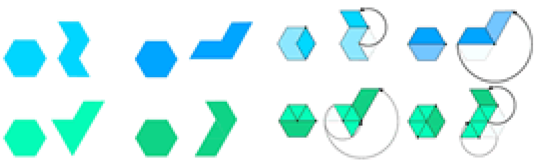


Ask for another, then another, then another...way of doing this.

What if there are more than two pieces?

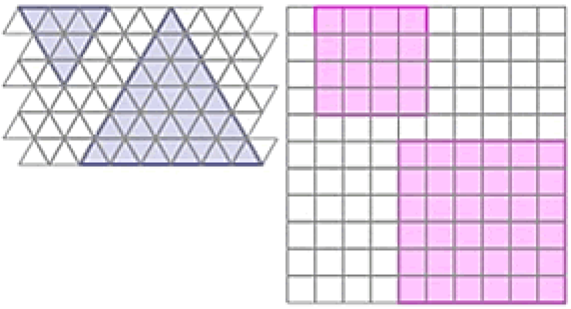


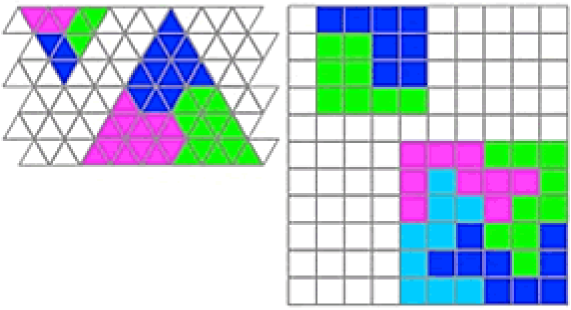
- Split a regular hexagon into *n* pieces of which *m* pieces are rotated to form a new shape. Vary the values of *n* and *m*.



Ask for another, then another, then another…example.

- Challenge your pupils to show how shapes-drawn-on-grids, such as the two grey triangles and two pink squares shown on the left of the diagram, can be split into parts that can be rotated onto each other. The two images at the top show one way (there are many possible ways!) of doing this to each of the four shapes at the bottom.

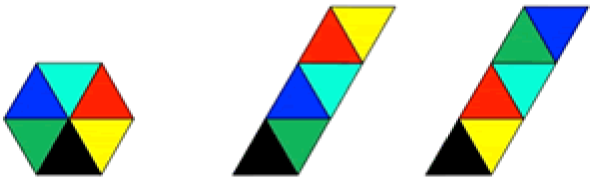




For each shape ask for another, then another, then another … .

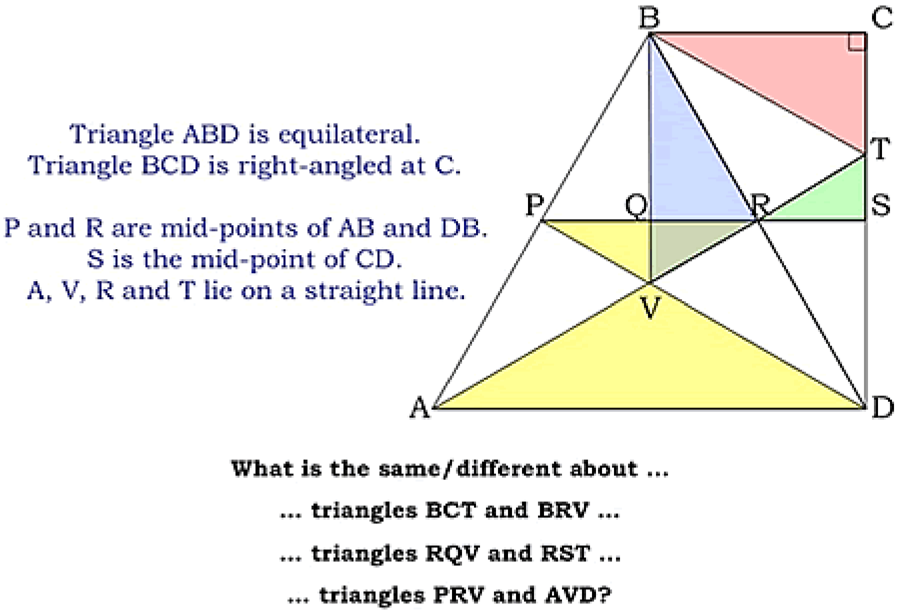
Same and Different

When it is possible to rotate pieces of a shape, such as six equilateral triangles composing a regular hexagon, in different ways to form the same new shape, you can ask "what is the same and what is different?" about the two methods and results. For example, in each case, about which points, and through which angles, are particular triangles in this hexagon rotated to form the parallelogram shown?

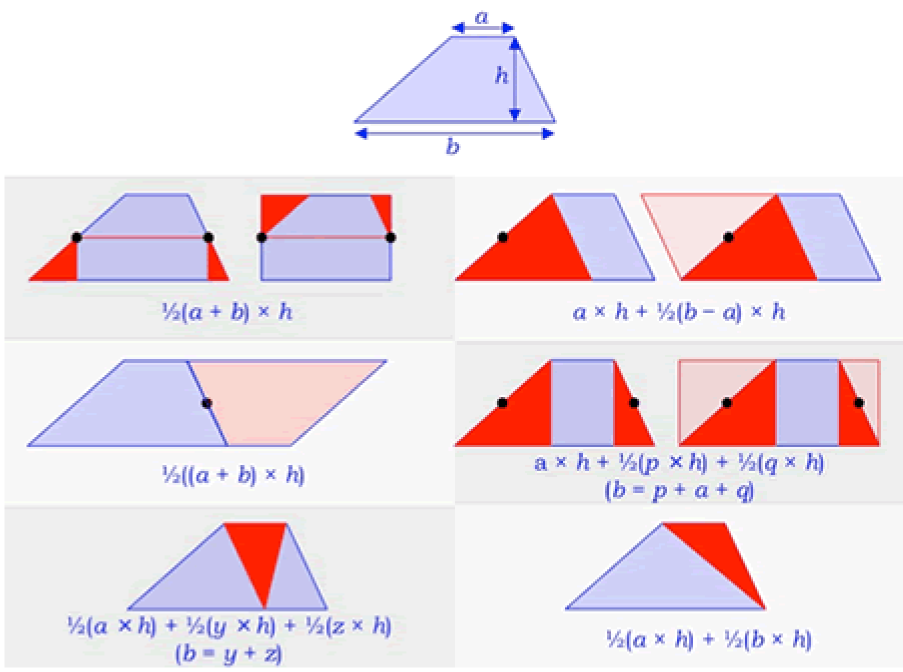


Pupils should come to see that labelling points and shapes helps them to describe which triangles are rotated, and how they are rotated. Diagrams showing how each parallelogram can be obtained by rotating parts of the hexagon are here.

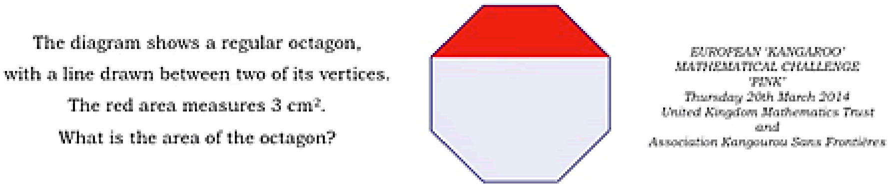
Deciding whether two objects are congruent by testing whether each can be mapped on to the other by a rotation is a powerful strategy in geometric reasoning. Struggling to describe, and explain how they know, what is the same and what is different about components of an image such as the one below will help pupils identify congruent shapes.



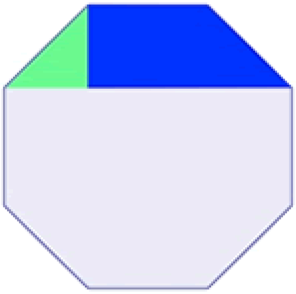
Some facts that pupils know can be proved visually using rotation. For example, there are many different ways that pupils can use valid reasoning to prove the formula giving the area of a trapezium, each of which corresponds to a different way of seeing the structure of a general trapezium. Looking for justifications for the trapezium area formula would make a good focus for an *Another and Another* challenge *(click image to enlarge)*:



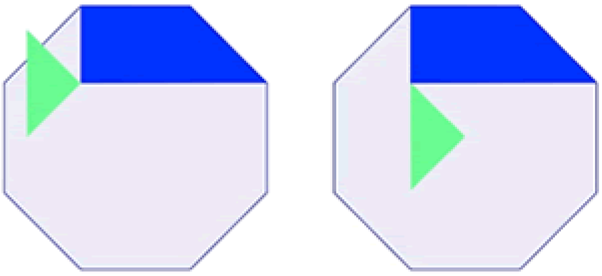
Pupils are sometimes completely stuck on a problem UNTIL they visualise a rotation. For example, consider this question from the UKMT website *(click image to enlarge)*:



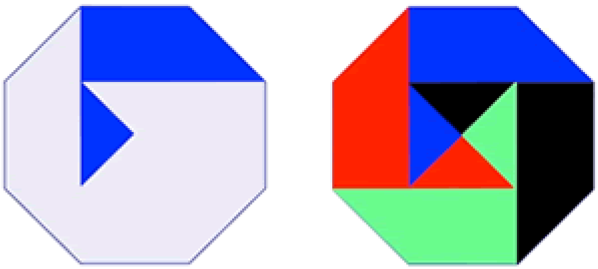
The solution can be seen, and explained, using only rotation and translation. First, visualise the red trapezium split into an isosceles right-angled triangle and a right-angled trapezium:



Now imagine rotating the green triangle 45˚ anti-clockwise about its bottom right-hand corner, and then translating it:

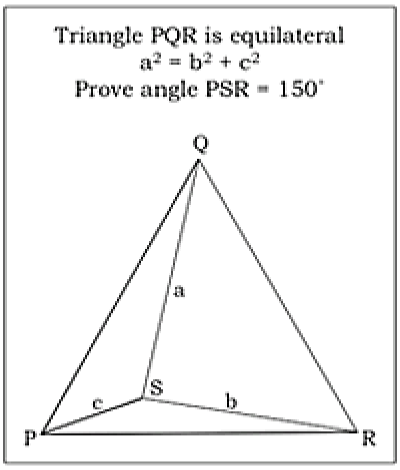


Mentally re-combine the triangle and small trapezium, and you can see that the shape that is formed can be repeatedly rotated through 90˚ to cover exactly the whole regular octagon:



So you literally see that the area of the whole octagon is four times the given area (3 cm^2) of the red trapezium. Therefore the area of the whole octagon is $3 \times 4 = 12$ cm^2.

You might like to offer your pupils this challenging problem from gogeometry.com. It can be proved using only rotations and one translation, and knowledge of the converse of Pythagoras' Theorem. Enjoy!



Let us know (info@ncetm.org.uk or via Twitter @NCETMsecondary) if you try any of these strategies: the examples you use, the responses your pupils create, and the impact and benefits you notice.

External links

Utah State University has a useful interactive application for exploring rotations with which users can design 'objects' (shapes and combinations of shapes), choose and change the rotation angle, move the centre of rotation, move/group/delete their 'objects', and turn the background square lattice into a coordinate grid with the origin wherever they want it to be.

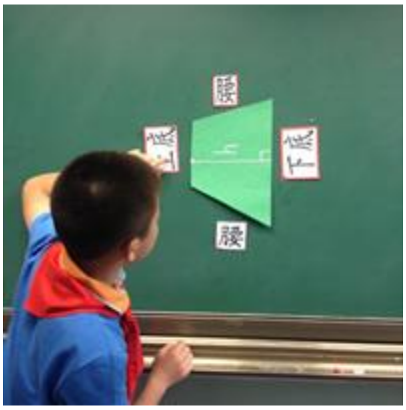
The animation Notes sur un triangle from the National Film Board of Canada is well worth watching and studying. Pupils could be challenged to describe, discuss, and possibly try to reproduce in drawings, small parts of it.

You can find previous *It Stands to Reason* features here

Image credit
Page header by Japanexpertna.se (adapted), some rights reserved

Eyes Down…

These two pictures show conceptual variation being used in a lesson to embed understanding. The girl pupil has labelled correctly the "top side" and the "bottom side" of the trapezium, and the boy pupil then explains to the class that the labels don't change position when the trapezium is rotated: the "top side" isn't the "top" side. The variation is the position of the trapezium; the concept being embedded is that the labelling is intrinsic (it's defined by the parallel sides of the trapezium) not extrinsic (it's not defined by the orientation of the trapezium relative to an observer). What other geometrical examples akin to this could there be? What examples of exemplifying conceptual variation in other strands, such as in algebra or proportion, could there be?



If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb) to us at *info@ncetm.org.uk* *with 'Secondary Magazine Eyes Down' in the email subject line. Include a note of where and when it was taken, and any comments on it you may have. If your picture is published, we'll send you a £20 voucher.*

You can find previous *Eyes Down* features here

Image credit
Page header by Kurt Bauschardt (adapted), some rights reserved