Secondary Magazine 115

National Centre for Excellence in the Teaching of Mathematics

Now the term is half way through – with no Bank Holiday in sight in either direction! – and the trees are losing leaves fast, this issue of the Secondary Magazine is full of sparkling ideas to brighten up Maths throughout November. This is the second edition of our refreshed Secondary magazine; let us know what you think, by email to info@ncetm.org.uk or on Twitter, @NCETMsecondary.

Contents

Heads Up

Whilst not wanting to define this article as mathematical gossip, it does come close! We've brought together news and current mathematical affairs, all in one place. We do hope it will interest you.

Happy Birthday Martin Gardner: 21 October 1914 – 22 May 2010. If you need a reason to recommend his books to your students, or use one of his brain-tingling puzzles to stimulate their reasoning skills, or share with them some of his erudite yet accessible writing, now you've got one!

Did you miss the recent challenge to build a MegaMenger? If you did, you can still build your own Menger Sponge with the downloadable Menger Cards, and you should also be able to catch up on the locations around the world taking part, as mentioned in Issue 113.

The new mathematics GCSEs for first teaching in 2015 have recently been accredited. The specifications are available from:

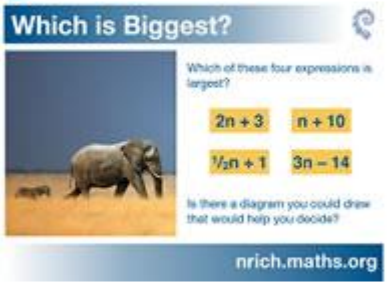
- AQA
- Edexcel
- OCR
- WJEC.

NSPCC Number Day will take place on 5 December 2014. Register now to involve your pupils in a national fundraising day that raises the profile of mathematics.

Applications are being accepted until 1 December for the Goldsmiths' Grant for Teachers: funding of up to £3000 is available plus a maximum of £2000 supply cover.

Have you found out about the new range of live online professional development opportunities available from the Further Maths Support Programme? You can engage in top quality CPD from the comfort of your own living room!

You may have been teaching when Radio Four's In Our Time programme was first broadcast on 25 September. Melvyn Bragg and his guests discussed Euler's number e: you can listen again or download the podcast. As the website states: *Euler's number, also known as e, was first discovered in the 17th century by the Swiss mathematician Jacob Bernoulli when he was studying compound interest. e is now recognised as one of the most important and interesting numbers in mathematics. Roughly equal to 2.718, e is useful in studying many everyday situations, from personal savings to epidemics. It also features in Euler's Identity, sometimes described as the most beautiful equation ever written.*



Two sets of colourful free posters to decorate your classroom are available, one from NRICH, and another from FMSP.

If you are interested in coincidences, you may have read about Robert May's School in Odiham, Hampshire, where teachers are struggling to put the right names to faces because there are nine sets of twins in the same year. Three sets of male twins, five sets of girls and one set of brother and sister are among the special collection. You could also think about coincidences using this 'double yolker' resource. Of course, the village of Midwich experienced something similar but altogether more sinister, in John Wyndham's 1957 sci-fi classic, The Midwich Cuckoos!

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Building Bridges
Adding, subtracting and multiplying fractions

In last month's article we looked at some of the complexity that lies under the surface of the seemingly simple notation of a fraction, in particular that a written fraction such as "$\frac{1}{3}$" can represent a stand-alone number (a point on a number line) as well as specifying the process "take a third of". Until secondary pupils are comfortable with the stand-alone number meaning of a fraction, they – understandably – find it hard to understand how it is even possible to carry out arithmetic with fractions, let alone do so confidently and fluently. Numbers are added, not processes; it doesn't make sense to say that you're adding the processes "take a third of" and "take a half of". The stand-alone number meaning must be secure before moving onto arithmetic; some ideas for embedding understanding of this were discussed in Issue 114.

Because this article comes under the *Building Bridges* strand, I am going to suggest ways of introducing adding, subtracting and multiplying with fractions that link to other parts of the maths curriculum. These links might suggest starter activities in your lesson(s) on fractions and arithmetic, or "warm-up" lessons in advance of them, so that the conceptual understanding of the arithmetic can be developed in advance of the technical procedures that need to be taught and practised.

Adding and subtracting: measurements with different units

If asked "what's £3 plus 3p?", almost all secondary pupils will know that the answer is not "6" of any unit, and will be able to explain why the answer in fact is £3.03 or 303p. Key to this is converting from one unit (£) to another (p), and this skill can be practised. UK currency soon becomes straightforward, whereas …

… In the country of Ruritania, there are six coins:

1 zollar 2 zags 3 zegs 4 zigs 6 zogs 12 zugs

And then you can ask "How many …

…zigs are worth the same as 5 zigs and 4 zags?
…zegs are worth the same as 5 zegs and 8 zugs?
…zugs are worth the same as 4 zegs and 5 zigs?

and "I go to the shop and buy something costing 2 zollars. I give the shopkeeper 5 zogs and 3 zigs. What change do I get?"

Pictures or actual objects – forged Ruritanian banknotes, for example! – could be useful here, so that pupils can physically exchange 2 zollar notes for 4 zag notes, or draw a line down the middle of a 1 zollar note to give two half-notes each worth 1 zag.

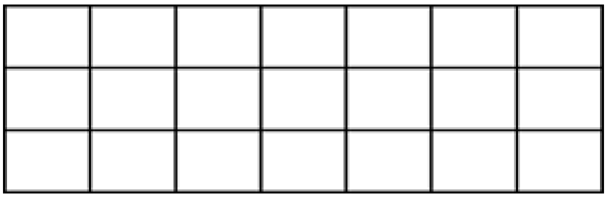
The transition to arithmetic with fractions comes by modelling the denominator as a unit of measurement, and by writing "3 quarters" rather than "$\frac{3}{4}$". Then 3 quarters and 2 fifths becomes the same as a money sum, evaluated by converting into the common currency unit of twentieths:

3 quarters and 2 fifths
= 15 twentieths + 8 twentieths
= 23 twentieths
= 1 whole and 3 twentieths.

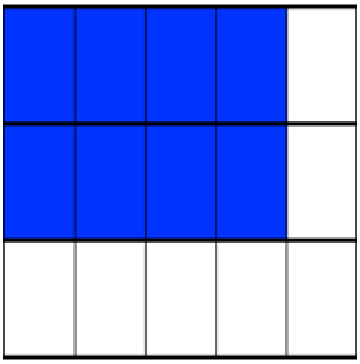
Again, physical banknotes, which can be drawn on or cut up, will help here, especially to identify the common currency unit. A key advantage of modelling the denominator as a unit – here, of currency but of course all this translates into units of length, mass, etc. – is that it removes the question “do we add the denominators?”: £3 + £5 isn’t ££8, and 3 sevenths + 2 sevenths isn’t 5 fourteenths (or 5 sevenths sevenths). The currency units aren’t added, so the denominators aren’t added; the same argument applies when subtracting.

Multiplying: arrays

Most pupils will be familiar from KS1 and 2 with arrays modelling multiplication:

first represent 3×7 as 

and then the answer is represented by the number of (small) rectangles created: 21. Units make explicit that this an area model for multiplication: $3\text{cm} \times 7\text{cm} = 21\text{cm}^2$. The area model can then be extended, so that $\frac{2}{3} \times \frac{4}{5}$ can be represented as

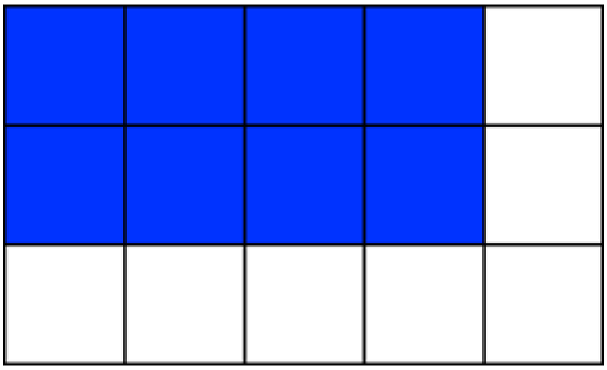


and the answer is the shaded area out of the total: 8 out of 15. Thinking of an array makes it clear why the algorithm is to multiply the denominators and to multiply the numerators: the product of the denominators is the total number of rectangles in the shape, and the product of the numerators is the number of shaded ones. The extension to the product of three fractions is obvious – and to four and more, sort of!

The strength of this model for multiplication is the strong connection with a model that pupils have seen previously. There are two weak spots, one procedural and one conceptual, which need keeping in mind.

1. This model doesn’t suggest cancelling before multiplying, so the products can be large and unwieldy. Understanding the cancelling of the denominator of one fraction with the numerator of the other has to come from exploration of the inverse relationship between the operations of multiplication and division (the argument that, for example, $\frac{2}{3} \times \frac{3}{5} = (2 \div 3) \times (3 \div 5) = 2$ **$\div$ 3 $\times$ 3** $\div 5 = 2 \times 1 \div 5 = 2 \div 5 = \frac{2}{5}$, because the operations “**$\div$ 3**” and “**$\times$ 3**” are an inverse pair equivalent to the operation “$\times$ 1”). The risk is that pupils’ development of procedural fluency is hindered if they don’t remember to cancel before multiplying.

2. Strictly, we're calculating $\frac{2}{3}$ of the **width** $\times$ $\frac{4}{5}$ of the **length** and saying that the shaded answer is an area that is fraction of the whole area. Conceptually, therefore, the width and length must BOTH be thought of as 1 (cm) – that's why the picture is a square, **not** a rectangle:



These weaknesses don't leave the model fatally flawed, but they might need raising and discussing.

To multiply mixed numbers, we can adapt the grid method used for long multiplication: just as 19×23 can be represented and evaluated as

x	10	9	
20	200	180	
3	30	27	
	230	207	437

so $2\frac{2}{3} \times 3\frac{1}{2}$ can be represented and evaluated as

x	2	⅔	
3	6	3 x ⅔ = 2	
½	2 x ½ = 1	½ x ⅔ = ⅓	
			9⅓

- as long as pupils are confident with the individual multiplications: "3 lots of 2 thirds is 6 thirds, which is 2 wholes" would be an explanation that suggested good conceptual understanding. This isn't as efficient as converting the mixed numbers to top-heavy fractions, but the link to prior learning is powerful.

Next month we'll look at models for division of integers, and then consider how to build a bridge from them to the division of fractions.

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Sixth Sense
Plaiting The Strands

Complex numbers and matrices are two of the key ideas in Further Pure Mathematics, both at AS and A2. They have deep connections to each other, but are often taught separately with only fleeting opportunities for students to think about them in a joined-up way. The card matching activity available with this article allows a number of strands of mathematics to be plaited together, and also paves the way for the future so that students get into the habit of looking for new connections all the time, and especially when they revisit these two topics.

Any complex number z = x + iy can be represented by the point (x, y) on a 2-D co-ordinate diagram, usually referred to as an Argand diagram in this context. Immediately we have a way of visualising real, imaginary and complex numbers on the same representation, and we can start to see direct links between complex numbers and the co-ordinate geometry of points, lines and circles studied first at GCSE then in AS Mathematics. Alternatively, any complex number z = x + iy can be represented by the vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Treated as a position vector relative to the origin O, this gives another way of describing the point representing z on the Argand diagram: the complex number z at the "tip" of the position vector z. However, treated as a free vector, we also have a way of visualising addition of the complex numbers z and w: we start the free vector representing z on the Argand diagram at the "tip" of the position vector representing w. Adding z to w can then be thought of as a translating w by z on the Argand diagram, which links back to work in GCSE Mathematics, and connects yet again arithmetic and geometry.

2 x 2 matrices can be used to represent (amongst other transformations) enlargements and rotations in the 2-D plane. The matrix $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ represents an enlargement centre the origin O with a scale factor of r, and the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents a rotation centre the origin O through θ° anticlockwise. This extends GCSE Mathematics work into AS Further Mathematics: students love that they can "do" transformations without drawing anything! The matrix $\begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ represents the combined transformation of the enlargement followed by the rotation, or the other way round. This gives a good opportunity to consider the commutativity of matrix multiplication and transformation composition: this pair combines commutatively (students should explain – geometrically – why) but is not generally true when multiplying 2 x 2 matrices (and students should suggest pairs of transformations that they can see won't combine commutatively).

Finally, we can associate any real number r with the matrix $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ and any complex number z = cos θ + i sin θ on the unit circle (see Secondary Newsletter 114) with the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then we can think of multiplication by a real number r as an enlargement (centre O, scale factor r) on the Argand diagram and multiplication by cos θ + i sin θ as a rotation (anticlockwise through θ° about O) on the Argand diagram. More generally, any complex number z = x + iy = r(cos θ + i sin θ) where r = $|z|$ and θ = arg z can be associated with the matrix $\begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$. Then we can think of multiplication by a complex number as a spiral symmetry, namely the combined transformation described above: an enlargement (centre O, scale factor r) followed by a rotation (anticlockwise through θ° about O) on the Argand diagram, or vice versa.

This card matching activity establishes links between complex numbers, coordinates, vectors, transformations and matrices, and will get students looking for further connections between different strands of mathematics as they move forward in their studies. Could your students add cards in the future to these to increase the connections (having studied loci in the Argand diagram, for example, or eigenvalues and eigenvectors), or make up similar sets of cards to display sets of connections in other areas of the syllabus they're studying?

Image credit

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From the Library
Shh! No Talking!

This new regular feature will highlight an article or research paper that has a bearing on teaching mathematics today

In this issue, It Stands to Reason focusses on pupils reasoning with fractions so our library article this month is Key understandings in mathematics learning, Paper 3: Understanding rational numbers and intensive quantities by Terezinha Nunes and Peter Bryant.

This paper is one of a series that were commissioned by the Nuffield Foundation in 2007 to review the available research literature on how children learn mathematics. The paper concentrates on how fractions are taught in primary schools, but we've suggested it for this month's library article because we think it will help you understand some of the difficulties your pupils may be having with fractions in KS3 and 4. As the article says,

> *"even after the age of 11 many students have difficulty in knowing whether two fractions are equivalent and do not know how to order some fractions. For example, in a study carried out in London, students were asked to paint ${}^{2}/_{3}$ of figures divided in 3, 6 and 9 equal parts. The majority solved the task correctly when the figure was divided into 3 parts but 40% of the 11- to 12-year-old students could not solve it when the figure was divided into 6 or 9 parts, which meant painting an equivalent fraction (${}^{4}/_{6}$ and ${}^{6}/_{9}$, respectively)."*

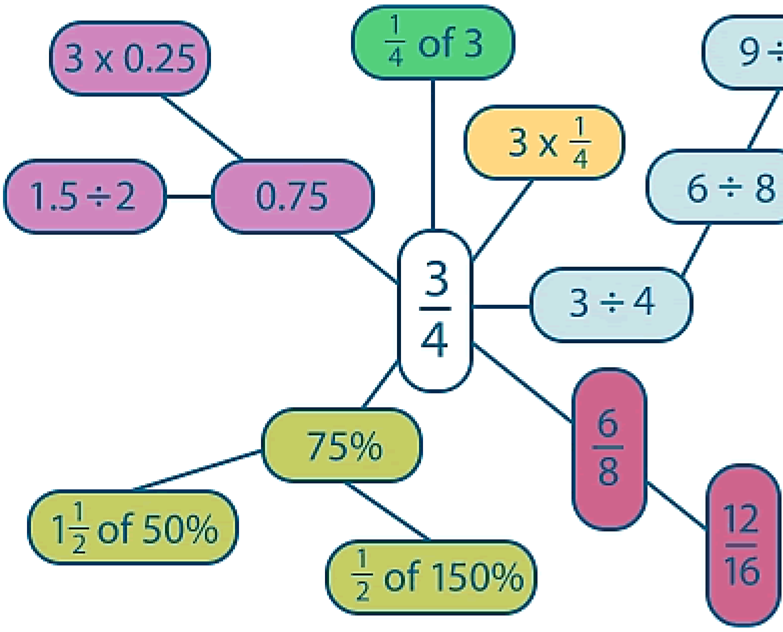
We think that, having read this paper, you will have a deeper understanding of why your pupils find fractions difficult, and also some ideas how to respond to this and to lead them to more secure conceptual and procedural confidence. Let us know what you think.

Image credit

It Stands to Reason

There's the old adage: a lost traveller asks for directions from a local resident, only to be told "well, to get there I wouldn't start from here". Teaching fractions in Years 7 and 8 can sometimes feel the same – but your pupils are where they are even if that's not where you want them to be. If so, it's well worth taking time to understand what they do and don't understand about fractions, and what they can and can't explain, and in this issue we share some fun ideas how you could do so.

Have a look at this spider diagram. You may like to add some things to a copy of the diagram yourself (click the diagram below to download as a PDF). See how many other links you can make:

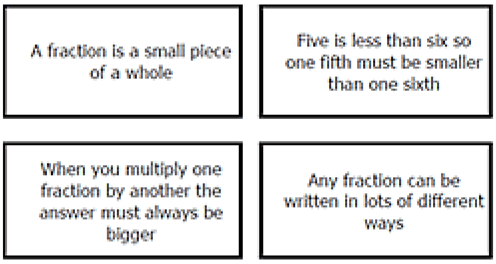


Now try this with your pupils. How many of these links do they make? Within the diagram you can see different meanings of fraction notation as both names of numbers and as operators. Fractions can be interpreted as:

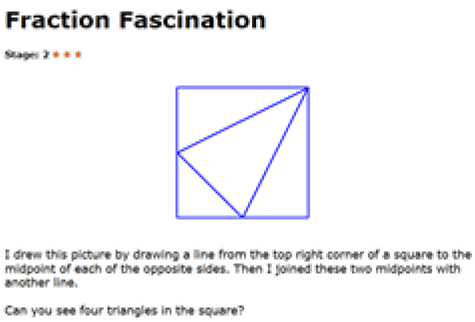
- part of a whole unit
- comparisons between part of a set and the whole set
- a point between two whole numbers
- result of a division operation
- comparing the sizes of two
  - sets of objects
  - measurements

It is the necessity of understanding that a fraction can be used in these different ways, and the need to work out which way that a particular fraction is being used in a given context, that makes fractions a hard concept for many pupils.

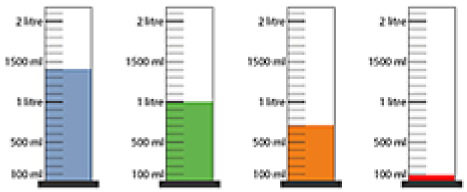
This Always, sometimes, never activity helps pupils by consolidating their ideas about what fractions are:



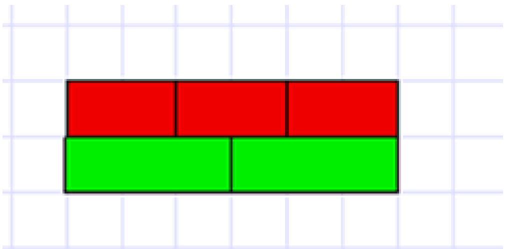
The Fraction Fascination task from NRICH encourages pupils to reason with the sizes of different fractions within a unit whole. The second activity is a good challenge.



Another activity that promotes reasoning skills is Oh Harry, where pupils are encouraged to link fractions to measurements of quantities. You might want to adapt the context to something a bit more "grown up"!

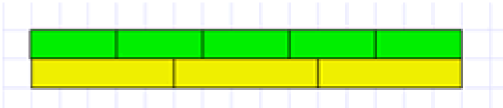


A key use of fractions is to make comparisons. Could all your pupils look at this diagram and describe the relationship between the two lengths?

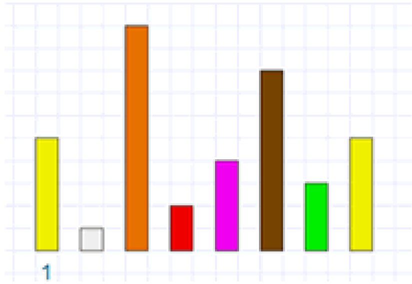


(Rich suggestions would be "The red length is two thirds of the green length" and "The green length is three halves of the red length").

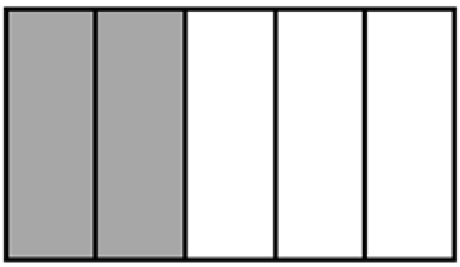
And in this diagram?



In this video clip, part of the microsite The teacher as researcher/teaching as researching, one teacher writes and talks about her work with Cuisenaire rods inspired by the writings of Gattegno and Goutard. She uses fractions to compare the lengths of the different colours. Pupils can be given rods and asked to suggest relationships between the rods, or given some relationships and asked to draw the rods that represent the relationships – the challenge is to do so with as few rods as possible.



This idea of comparison can be explored further by asking pupils what fractions can they see in this diagram:



Your pupils will probably identify $\frac{2}{5}$ and $\frac{3}{5}$ but note if they say "$\frac{2}{5}$ of the whole" or just "$\frac{2}{5}$". Does anyone say something of the form "x is $\frac{2}{3}$ of y" or "x is $\frac{3}{2}$ of y"? Can such a statement be explained clearly?

Activities such as this one make good starters or mid-lesson revitalisers, and will help you ensure that your pupils are all starting in the right place for the journey ahead of them. There are further resources here:

- some ideas about fractions themselves and issues related to teaching fractions can be found in the NRDC booklet, Fractions, by Rachel McLeod and Barbara Newmarch
- this NRICH article, Understanding Fractions, which includes links to other activities
- this NCETM Departmental Workshop, Fractions.

Image credit

Eyes Down

This feature will be a regular picture-based item that you might use with your pupils, or your department, or just by yourself, to make you think about something in a different way

This picture was taken at Barrington Court, a National Trust property in Somerset - another holiday snap!

This stunning sundial has a gnomon on ten of the twelve faces. Many sundials only have one dial from which to read the time and this tends to be a flat dial in either a horizontal or vertical plane. These ten dials make it possible to tell the time for a greater span of the day - as long as the sun is shining!

You could ask your pupils:

- How many faces, vertices and edges does the polyhedron around which the sundial is built have?
- What shape is each of its faces?
- What other polyhedra have faces all the same shape? (The Platonic solids).
- Every polyhedron has a dual, in which the vertices of the dual polyhedron are the midpoint of the faces of the original polyhedron. What are the dual polyhedra for the platonic solids?

And then they can download and make their own dodecahedron calendar for 2015.

If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb), and a short (150 words maximum) commentary, to us at info@ncetm.org.uk - we look forward to hearing from you!

Image credit