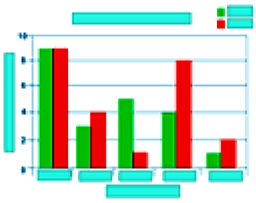
Secondary Magazine 146

National Centre for Excellence in the Teaching of Mathematics

Welcome to the February edition of our Secondary Magazine, with longer days, and the prospect of seeing some daylight outside school hours! In this edition, we take the opportunity to visit a couple of the many Maths Hubs Network Collaborative Projects, being run by local Work Groups of teachers at a Maths Hub near you.

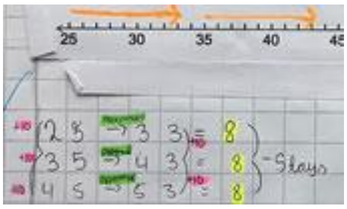
Don't forget that all previous issues are available in the Archive.

This issue's featured articles



Teachers Thinking about Mathematical Thinking

First, we visit the 'Mathematical Thinking and the new GCSE' Work Group, at the East Midlands East Maths Hub. As well as providing a flavour of the work being done by the group, and of the depth of discussion, we show some of the ideas being tried out by the teachers in their own classrooms and consider how their pupils attempt reasoning tasks.

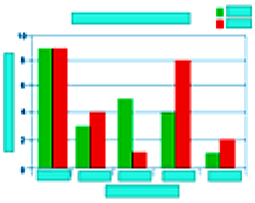


Adding Meaning to Subtracting

Robert Wilne, leading the 'Improving Continuity Across the Y5-8 Transition' Work Group for the London Thames Maths Hub, reflects on the conceptual models of subtraction that we offer learners of maths. His article is inspired by an episode with his Y11 Higher Tier group, when he unexpectedly uncovered cracks in the foundations of their understanding, and also by his work with the project. He suggests models of subtraction that he believes should be taught in a continuous way throughout primary and secondary school.

And here are some other things for your attention:

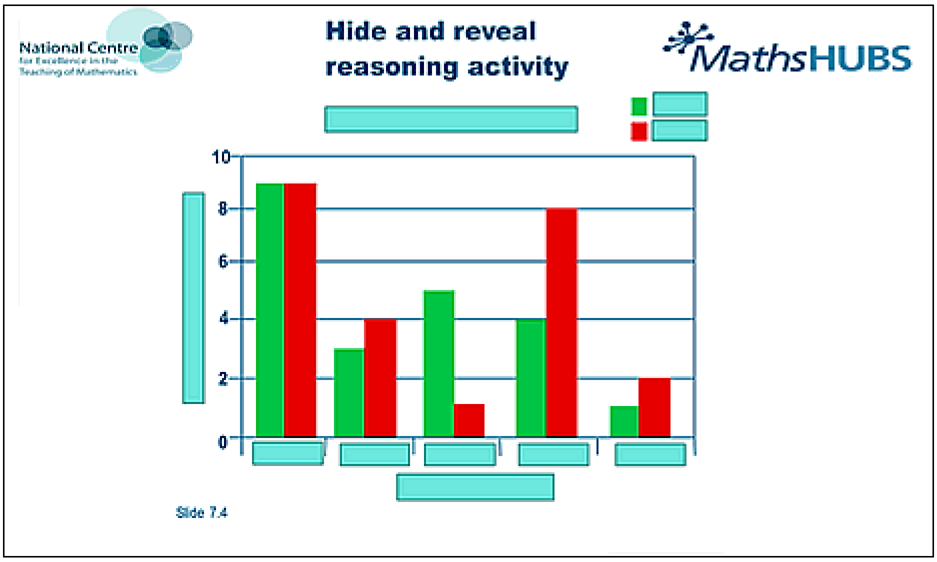
- With renewed and enthusiastic backing from the DfE, the NCETM and Maths Hubs have launched recruitment for new cohorts of Mastery Specialists in both primary and secondary phases. Find out more, including details of how to apply. The application deadline for the secondary phase is **5pm on Monday 19 February**.
- Continuity of fractions learning from Year 5 to Year 8 is the topic handled in the latest NCETM podcast.
- In case you missed this episode of BBC's *The Life Scientific*, it's well worth listening to. Eugenia Cheng not only oozes a passion for maths, but also gets under the skin of why young people - in particular girls - may be turned off the subject.
- Do you have an exceptional and ambitious student who will be 16+ by the summer? Promys Europe offers a six-week residential opportunity based at Oxford University, for a small number of secondary school students to explore deeply the creative world of mathematics. Full/partial financial aid is available. Applications close on **11 February**, so hurry!
- MEI has recently updated the curriculum mapping of its 130+ Maths Item of the Month problems. Covering a wide range of KS4 and KS5 topics, they are great resources to promote mathematical thinking, proof and problem solving.
- Cambridge Assessment are researching how the current reforms to AS/A levels are affecting schools/colleges. Heads of Department, please consider completing their survey. For 15 minutes of your time, there is the chance to win a £100 book token. Closing date is **1 March**.



Teachers Thinking about Mathematical Thinking

This is an account of a Work Group being run by the East Midlands East Maths Hub, but this is one of 35, running in every Maths Hub across the country as part of a Maths Hubs Network Collaborative Project. To get involved, contact your local Maths Hub.

"What could be on the hidden labels?"



The maths teachers present, all very familiar with graphs of this type, think for a while, and some lively discussion ensues.

"I'm not going to show you all the labels. Why not? Because getting it right or wrong is not the point here – it's the discussion we are having that's important."

So says James Thomas, Work Group Lead for East Midlands East Maths Hub, leading the first session of the Work Group: 'Mathematical Thinking and the new GCSE', with a small group of teachers from local schools.

What would *your* GCSE students make of this hide/reveal task?

James admits that if he was as evasive about the 'reveal' part of this task, with his Y10 class, "there would be a riot", but his point is made – that learners of mathematics often get very tied up with whether they are 'right' or 'wrong'. This can be a crutch for those that are normally right, and equally powerfully, something that makes maths terrifying for others (and not only those that often get it wrong). In a task like this, James has taken a graph that you might find in any secondary maths textbook (usually with a set of closed questions alongside) and created a rich and meaningful discussion, where students have to engage with the data to argue why it could, or could not, represent the data suggested by their classmates.

Hide/reveal is just one of the powerful techniques that the Work Group is discussing, for promoting reasoning and problem solving in secondary classrooms. They form a toolkit with which teachers, even inexperienced or non-maths specialist teachers, can adapt standard textbook questions to discourage students from adopting a 'learn the algorithm' approach. The techniques are designed to adapt standard, and readily available material in a way that promotes deeper thinking and therefore more connected understanding. Teachers in the Work Group acknowledge that the content demands at GCSE means that reasoning and problem-solving can easily get 'tacked on' to an already full curriculum. There is a danger that it becomes the part of the curriculum that gets left out when time is short, or only accessed by higher-attaining students, rather than being integrated throughout learning. The toolkit provides a way to bring reasoning into every lesson.

Below, are a couple of examples of 'generic' approaches that can open up thinking on any topic, question or context:

Example 1: Sequencing Lines of Working Out / Here's the Calculation, What's the Question?

A shop sells flour from large sacks at a rate of £1.20 for 750g. As an introductory offer the shop is currently offering a discount of ___% on all sales of flour.

The working out a student used to answer a problem related to this information is shown mixed up, below.

$60 \div 5 = 12$ $\quad$ $14.40 - 3.60 = 10.80$ $\quad$ $12 \times 1.20 = 14.40$

$14.40 \times 0.25 = 3.60$

*Can you construct what the original problem might have been, and can you **rearrange** the 'working out' into the order you think the student might have had it?*

*Write **a reason** explaining the meaning of each line of working out in relation to the problem.*
*What was the **discount on offer?***

Example 2: Developing chains of reasoning and deepening understanding

Below is an Edexcel question from the June 2017 GCSE exam to which a number of the Work Group activities could be applied to open up opportunities for reasoning and deepening understanding. Where pupils may already be confident answering the question, we look here at opportunities to use the structure of the problem to develop further chains of reasoning.

18 Daniel bakes 420 cakes.
He bakes only vanilla cakes, banana cakes, lemon cakes and chocolate cakes.

$\frac{2}{7}$ of the cakes are vanilla cakes.

35% of the cakes are banana cakes.
The ratio of the number of lemon cakes to the number of chocolate cakes is 4:5

Work out the number of lemon cakes Daniel bakes.

On another day Daniel bakes a different number of vanilla, banana, lemon and chocolate cakes but the quantities remain in the same proportion.

Challenge:

1. If you knew how many vanilla cakes were baked could you work out how many banana cakes were baked? *Give a chain of reasoning (What about in reverse?).*

2. ***Try something more challenging***: if you knew the number of chocolate cakes baked could you work out the number of banana cakes?

Choose any two of the items below, how could you work one of them out if you know the other? Explain your chain of reasoning. *What other discussion issues might arise out of the answers?*

A) Total number of cakes baked	
B) Number of vanilla cakes baked	
C) Number of banana cakes baked	
D) Number of lemon cakes baked	
E) Number of chocolate cakes baked	

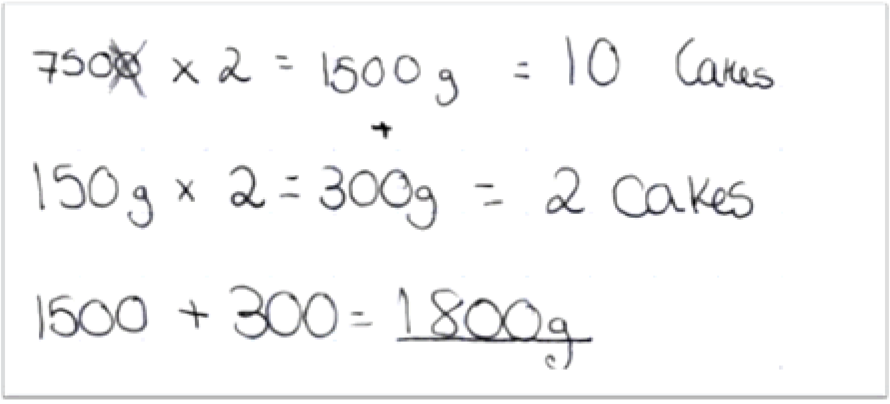
In the session we visited, the Work Group also spent some time considering 'what is reasoning?', and how does it differ from problem-solving? Using the NRICH article Reasoning: the Journey from Novice to Expert, that describes five stages of reasoning (Describing, Explaining, Convincing, Justifying and Proving), they considered some pupil work and tried to determine which of the NRICH stages best describes each piece. Participants in the Work Group concluded - perhaps unsurprisingly - that the pupils were able to reason better verbally than in written form. This highlights a challenge for teachers: to teach students to express what they can tell someone verbally, in effective written form for examiners.

The problem is fairly standard:

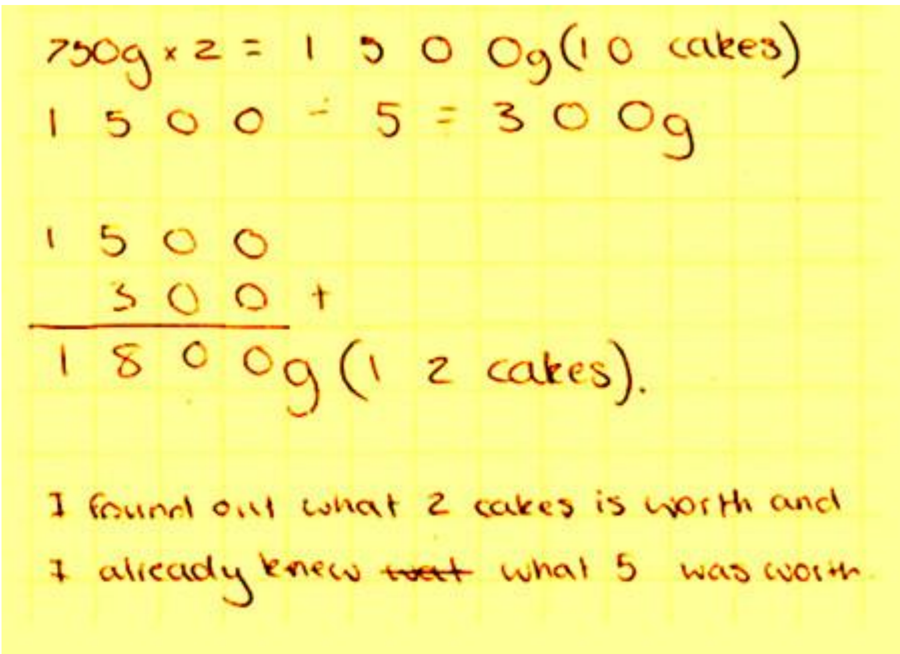
> A 750g bag of flour costing £1.20 provides enough flour to make 5 sponge cakes
>
> Find the weight of flour needed for 12 sponge cakes

What presents a challenge to pupils is expressing their reasoning: verbally, but also in written form.

Try assessing their reasoning yourself. Have a look at these two pieces of work. Consider first, the written argument– how complete is the reasoning? (You could use the NRICH scale). Then play the short video to hear the verbal argument – is it any more complete?



- Watch the video here

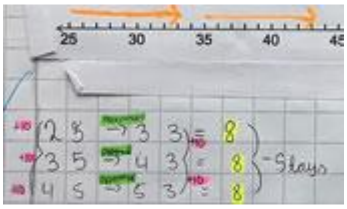


- Watch the video here

The Work Group that we visited is part of a Maths Hubs Network Collaborative Project, 'Mathematical Thinking and the new GCSE', being run by Maths Hubs throughout England. James is one of the 35 Work Group Leads facilitating the project in his own locality. Based on best practice researched and refined by the Multiplicative Reasoning at KS3 project run in Maths Hubs over the past two years, this national project aims to support long term development of skills throughout the secondary curriculum, as well as the immediate needs of GCSE students. The approaches are practical, accessible and classroom-based.

This model of professional development is designed to engender long-term, sustainable change in classroom practice, throughout the departments of participating teachers. When schools sign up a teacher, or preferably, a pair of teachers, they are committing to four days of release time, plus the time that the teachers need to lead their departments in introducing more reasoning into their classrooms. Between each Work Group day, there is a 'gap task', typically involving trying out one of the activities so as to be able to reflect deeply on the process with other teachers in the group, or using Lesson Study to observe a colleague using one of the activities. It's a commitment for departments that are looking for genuine, long-term improvement in how they teach reasoning and problem-solving.

If you are interested in taking part in this kind of professional development, contact your local Maths Hub now – there may still be opportunities this year, and certainly Hubs are keen to hear from teachers wanting to get involved in 2018/19.



Assessing Mastery at KS3

Robert Wilne is Deputy Director (Maths) at the Atlas TSA family of seven primary and secondary schools in South East London, where he is leading the development of their 3-19 maths curriculum. He's also the London Thames Maths Hub Work Group Lead for the 'Improving Continuity Across the Y5-8 Transition' Network Collaborative Project. You can hear Robert talking about the Y5-8 Transition Project (in the context of fractions) in our latest podcast. And you can read more about a continuous approach to fractions in KS2/3 in the latest issue of Bespoke, the newsletter of the Maths Hubs programme.

Last May I was teaching a Y11 Higher Tier revision class: 25 or so students with secure mock grades 7+. Then suddenly, in the middle of their confident answering of exam-style questions about the cosine rule, quadratic simultaneous equations and the like, it all crashed to a halt and I found myself drawing a number line and explaining to them about "9 – 5 = 4"!

I'd asked them

- The point P has coordinates (-3, -2) and the point R has coordinates (9, 18). Q is on the line PR, such that PQ is 1/3 the length of QR. What are the coordinates of Q?

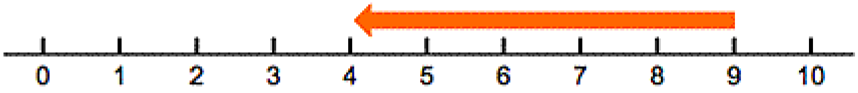
Pretty quickly, a student came to the board, drew a diagram and explained her reasoning:

> "9 minus minus 3 is 12 and 18 minus minus 2 is 20, and we want ¼ of each of these so it's 3 and 5, so we add 3 to minus 3 and 5 to minus 2 and get 0 and 3 for Q's coordinates."

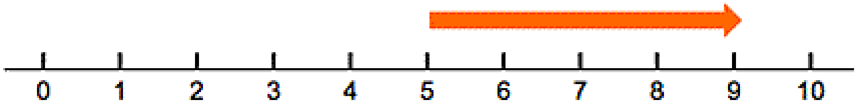
Her peers agreed, we sharpened the language (discussed in the footnote at the end of this article) and were about to move on when another student asked

> "Why are we minusing? We want to find the difference between -3 and 9, what's that got to do with subtracting?"

And that question revealed to me a huge gap in his (and in his peers' as they agreed with him) conceptual understanding of something as fundamental, as bottom-layer-of-the-Jenga-tower, as whole number subtraction: that '9 – 5 = 4' is an abstract representation of the two very different concrete processes of 'take away' and 'difference between'. The **same** string of symbols represents two very **different** pictures:



9 take away 5 is 4



The difference between 5 and 9 is 4

The pictures are different because the models are different: the first one is "subtraction representing reduction" and the second is "subtraction representing comparison". The different models always give

numerically the same answer, so long as we are precise about interpreting "difference ***between***" as "***from*** the subtrahend (the second number in the subtraction) to the minuend (the first)". This precision, that *difference between* means ***difference from*** the subtrahend ***to*** the minuend, becomes necessary when the abstract subtractions become less easy, or less natural, to interpret as 'take-away' in the concrete, particularly when the subtrahend is negative:

- Why does 9 subtract 3 = 6? Because the difference between 3 and 9, from 3 to 9, is 6.
- Why does -9 subtract 3 = -12? Because the difference between 3 and -9, from 3 to -9, is -12.
- Why does 9 subtract -3 = 12? Because the difference between -3 and 9, from -3 to 9, is 12.
- Why does -9 subtract -3 = -6? Because the difference between -3 and -9, from -3 to -9, is -6.

A model of temperature change leads to the same numerical answers:

- if the thermometer yesterday recorded 3°C and today records -9°C, then the temperature change from 3°C to -9°C is a fall of 12°C, which we can write as -12°C, hence -9 - 3 = -12;
- if yesterday it recorded -3°C and today it records -9°C, then the temperature change from -3°C to -9°C is a fall of 6°C, which we can write as -6°C, hence -9 - -3 = -6;
- if yesterday it recorded -9°C and today it records -3°C, then the temperature change from -9°C to -3°C is a rise of 6°C, which we can write as 6°C, hence -3 - -9 = 6.

My Y11 students didn't know, or didn't grasp, the difference between (pun fully intended) 'difference between' and 'taking away': why not? They've known about subtraction for umpteen years! The operation of subtraction occurs in every key stage: the youngest learners subtract small positive integers from slightly larger ones, and A-level Further Maths students subtract the arguments of complex numbers and wonder what the connection is with logarithms. In the middle, usually in Y7 or Y8, there is the conceptual shift from subtracting 'numbers of things' to subtracting 'abstract numbers', and then to subtracting 'abstract expressions'. KS3 teachers should be asking their students to explore what's the same and what's different about '9 – 3' and '9½ – 3.7', and '9 – -3' and '-9 – 3', and '9x – 3x' and '-9x – -3x'.

If learners are to develop confident, flexible and secure understanding of subtraction, they need to encounter it in a conceptually and procedurally continuous way, in every key stage. That can only happen if teachers in each key stage communicate with each other about how they are teaching subtraction: the procedures their students are using, and the language they are using to describe what those procedures are representing. For example, consider this Y6 SATs question:

19

Amina posts three large letters.

The postage costs the same for each letter.

She pays with a £20 note.

Her change is £14.96

What is the cost of posting one letter?

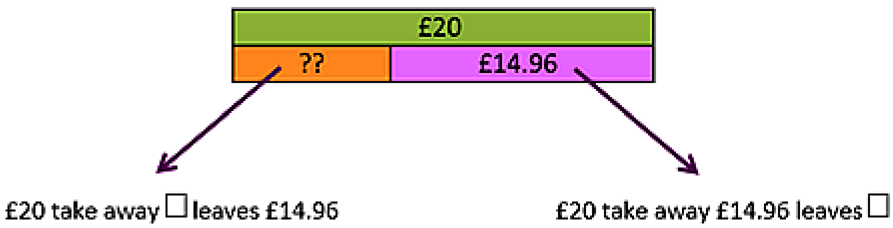
My Y11 students didn't know, or didn't grasp, the difference between (pun fully intended) 'difference between' and 'taking away': why not? They've known about subtraction for umpteen years! The operation of subtraction occurs in every key stage: the youngest learners subtract small positive integers from slightly larger ones, and A-level Further Maths students subtract the arguments of complex numbers and wonder what the connection is with logarithms. In the middle, usually in Y7 or Y8, there is the conceptual shift from subtracting 'numbers of things' to subtracting 'abstract numbers', and then to subtracting 'abstract expressions'. KS3 teachers should be asking their students to explore what's the same and what's different about '9 – 3' and '9½ – 3.7', and '9 – -3' and '-9 – 3', and '9x – 3x' and '-9x – -3x'.

If learners are to develop confident, flexible and secure understanding of subtraction, they need to encounter it in a conceptually and procedurally continuous way, in every key stage. That can only happen if teachers in each key stage communicate with each other about how they are teaching subtraction: the procedures their students are using, and the language they are using to describe what those procedures are representing. For example, consider this Y6 SATs question:

The first step is to calculate "£20 subtract £14.96". In my experience, most secondary teachers see this as a 'taking away' problem, and they expect their students to apply the column subtraction algorithm and get the answer "£5.04" (assuming they navigate (twice!) the difficulty of exchanging between adjacent columns). But many primary pupils see this as a 'difference between' problem, and they reason "£14.96 add on 4p and then add on £5 is £20, so the difference from £14.96 to £20 is £5.04". If the secondary teacher presents 'taking away' reasoning to a class of pupils who are used to 'difference between' reasoning, confusion is inevitable.

The primary pupils' 'difference between' reasoning is deeper and more powerful than might at first appear: they are reasoning, probably without realising, that

- this is a 'taking away' problem, because Amina had £20 and the shopkeeper took away some of it in exchange for stamps, and Amina was left with £14.96,
- so the problem we need to solve is "£20 take away ☐ leaves £14.96, how much does ☐ represent?"
- but this is conceptually the same as '£20 take away £14.96 leaves ☐, how much does ☐ represent?"

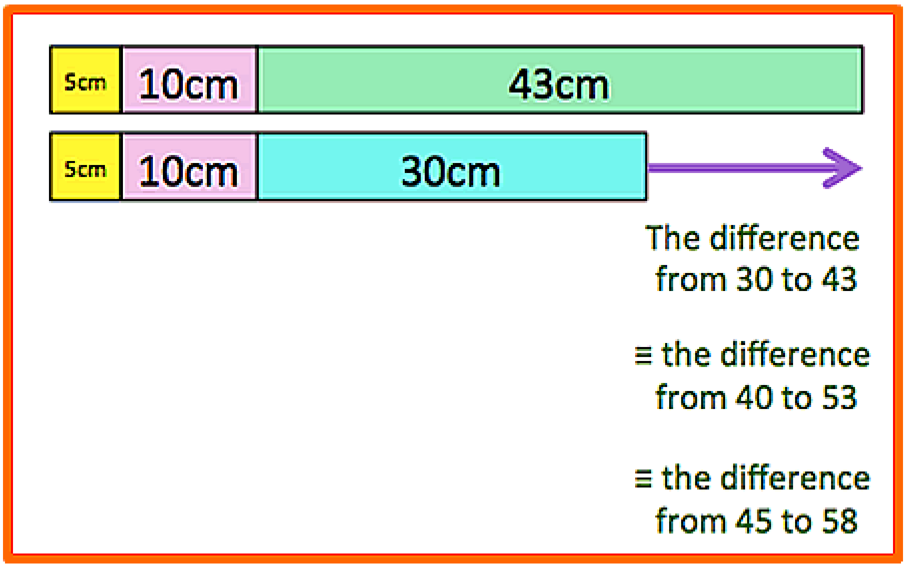


- and that is numerically but not conceptually the same as "the difference between £14.96 and £20"
- which I can work out by starting with £14.96 and augmenting that number by 4p and then £5 until I reach £20
- so the difference between £14.96 and £20, from £14.96 to £20 is £5.04
- so £20 take away £14.96 is also £5.04.

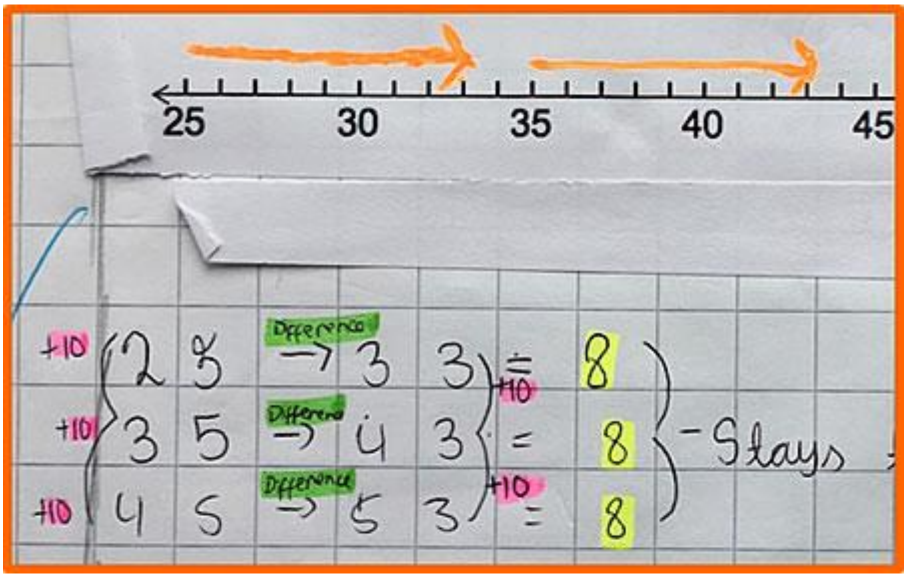
The pupils, therefore, are switching between models: they are working out a 'take away' by thinking of it as a 'difference between, from subtrahend to minuend'. Similarly, we can reason that:

58 'take away' 45
≡ 53 'take away' 40
≡ 43 'take away' 30

without working out that each subtraction equals 13:



This is **The Principle of Constant Difference**: in a subtraction, if the minuend and the subtrahend both increase or decrease by the same amount, the difference between them (from the subtrahend to the minuend) stays the same. This Y7 student's number line shows the 'difference between' reasoning clearly. She has drawn arrows of the same length to show that the difference from the subtrahend to the minuend is constant when they both increase by 10:

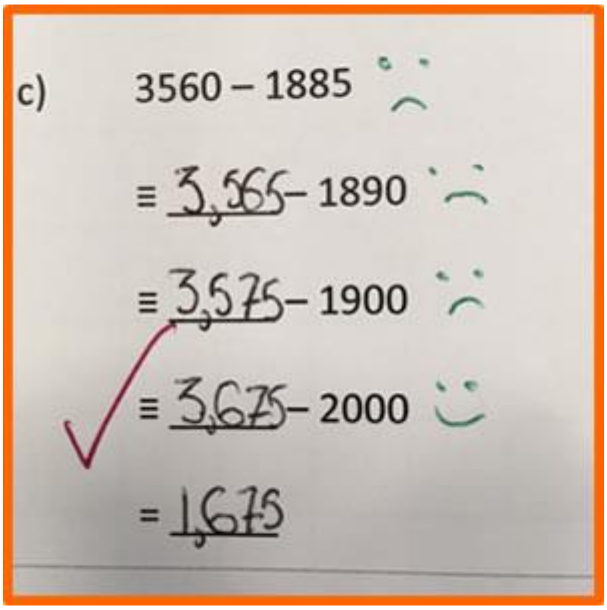


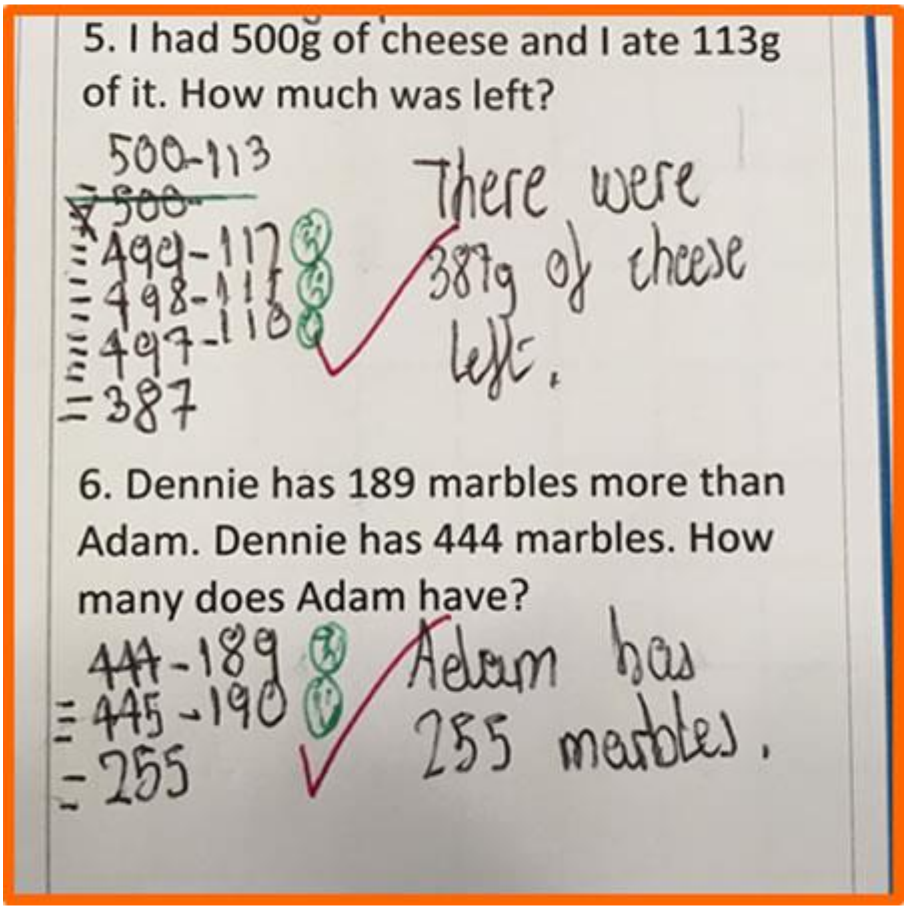
To return to Amina's stamps: the Principle of Constant Difference is an efficient way to calculate

- £20.00 – £14.96 'take away' has the same numerical answer as 'difference between'
- ≡ £20.01 – £14.97 because the minuend and the subtrahend both increase by 0.01
- ≡ £20.02 – £14.98 because the minuend and the subtrahend both increase by 0.01
- ≡ £20.03 – £14.99 because the minuend and the subtrahend both increase by 0.01
- ≡ £20.04 – £15.00 which is easy to work out: a 'nasty' subtraction has become 'nice'
- = £5.04

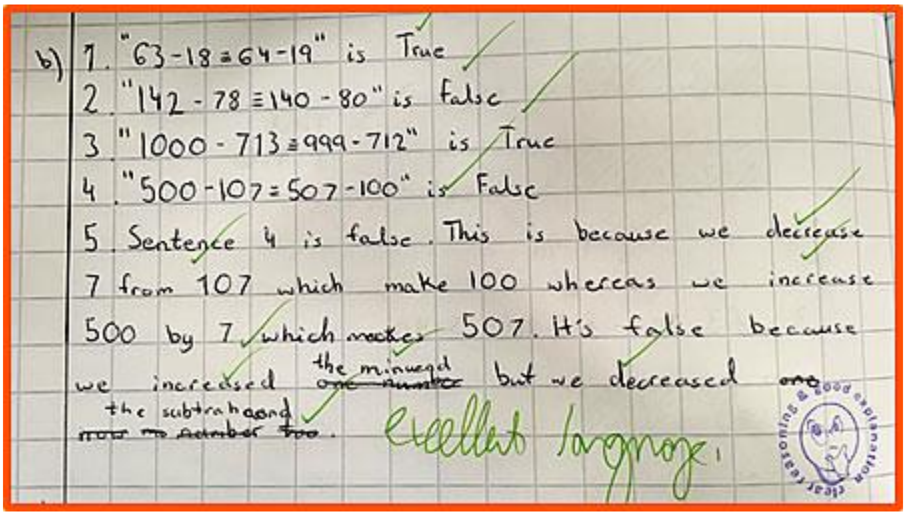
Pupils can use the Principle to reason about subtraction throughout KS2 and KS3:

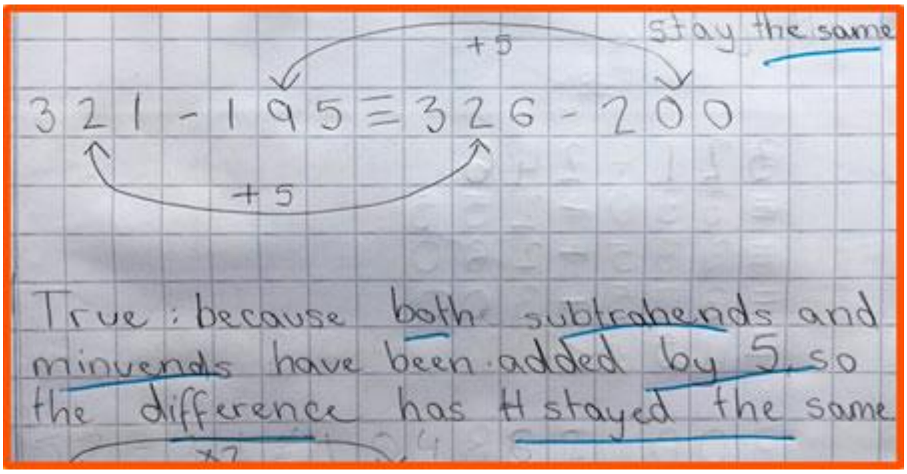
- in Y5




5. I had 500g of cheese and I ate 113g of it. How much was left?
500-113
= 499-112
= 498-111
= 497-110
= 387
There were 387g of cheese left.
6. Dennie has 189 marbles more than Adam. Dennie has 444 marbles. How many does Adam have?
444-189
= 445-190
= 255
Adam has 255 marbles.

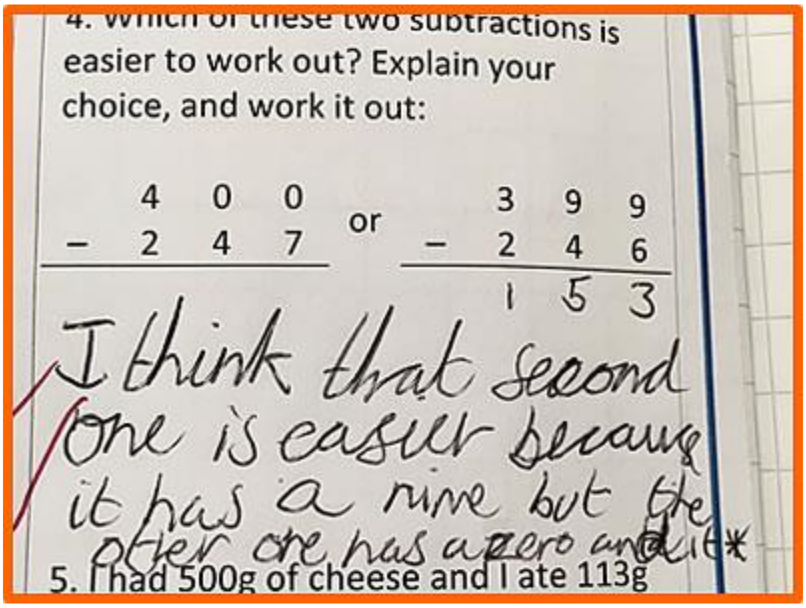
- and in Y7


b) 1. "63-18=64-19" is True
2. "142-78=140-80" is False
3. "1000-713=999-712" is True
4. "500-107=507-100" is False
5. Sentence 4 is false. This is because we decrease 7 from 107 which make 100 whereas we increase 500 by 7 which makes 507. It's false because we increased the minuend but we decreased the subtrahend
excellent language.
Clear reasoning & good explanation



Notice that the Y7 students are reasoning about the subtractions, rather than working them out explicitly: this activity is developing their **conceptual understanding** rather than their **procedural fluency**.

Similarly in Y5:



The power of the Principle is that it extends naturally to subtractions that are procedurally more demanding and/or conceptually more challenging, in particular subtractions with negative subtrahends, and then also those with algebraic terms in the minuend, the subtrahend, or both:

- 5.3 – 2.7 take away' has the same numerical answer as 'difference between'
- ≡ 5.6 – 3 because the minuend and the subtrahend both increase by 0.3
- = 2.6 which is easy to work out: a 'nasty' subtraction has become 'nice'

and then

- 8 – -2 take away' has the same numerical answer as 'difference between'
- ≡ 9 – -1 because the minuend and the subtrahend both increase by 1
- ≡ 10 – 0 because the minuend and the subtrahend both increase by 1
- = 10 which is easy to work out: a 'nasty' subtraction has become 'nice'

and

- -5 – -13 'take away' has the same numerical answer as 'difference between'
- ≡ -2 – -10 because the minuend and the subtrahend both increase by 3
- ≡ 0 – -8 because the minuend and the subtrahend both increase by 2
- 8 – 0 because the minuend and the subtrahend both increase by 8
- = 8 which is easy to work out: a 'nasty' subtraction has become 'nice'

and

- -3.8 – -7.39
- ≡ -4.0 – -7.59 because the minuend and the subtrahend both decrease by 0.2
- ≡ 0 – -3.59 because the minuend and the subtrahend both increase by 4
- ≡ 3.59 – 0 because the minuend and the subtrahend both increase by 3.59
- = 3.59 'nasty' has become 'nice'

and then, later still

- -9x – -3x
- ≡ -8x – -2x because the minuend and the subtrahend both increase by x
- ≡ -7x – -x because the minuend and the subtrahend both increase by x
- ≡ -6x – 0 because the minuend and the subtrahend both increase by x
- = -6x 'nasty' has become 'nice'

and

- $(3x + 5) - (x + 4)$
- $\equiv (3x + 1) - (x)$ because the minuend and the subtrahend both decrease by 4
- $= 2x + 1$ 'nasty' has become 'nice'

and

- $(7x - 3) - (2x - 5)$
- $\equiv (7x + 2) - (2x)$ because the minuend and the subtrahend both increase by 5
- $= 5x + 2$ 'nasty' has become 'nice'

or perhaps

- $(7x - 3) - (2x - 5)$
- $\equiv (5x - 3) - (-5)$ because the minuend and the subtrahend both decrease by 2x
- $\equiv (5x + 2) - (0)$ because the minuend and the subtrahend both increase by 5
- $= 5x + 2$ 'nasty' has become 'nice'

The Principle of Constant Difference is procedurally powerful and conceptually rich, and is simple enough that it can be – I would say it should be – explored and grappled with and used throughout the 'middle years', i.e. from Y5 to Y8. This would ensure that pupils do indeed encounter subtraction in a conceptually and procedurally continuous way from primary to secondary. For this to happen, though, primary and secondary teachers need to be

- confident themselves with justifying and using the Principle;
- confident that their 'feeder primary' or 'destination secondary' colleagues are justifying and using it too;

and for *this* to happen, there needs to be regular cross-phase communication between subject leaders – or, even better, cross-phase professional development – so that primary and secondary teachers come together and learn together, as recommended by the Education Endowment Foundation:

> **Are primary and secondary schools developing a shared understandings of curriculum, teaching, and learning?** Both primary and secondary teachers are likely to be more effective if they are familiar with the mathematics curriculum and teaching methods outside of their age-phase.

Creating the opportunity for, and the culture of, dialogue and development between primary and secondary teachers is one of the key aims of the Maths Hubs project 'Improving Continuity Between Primary and Secondary School'. To find out more and get involved, contact your local Maths Hub directly, which you can do via www.mathshubs.org.uk.

Footnote

"9 minus minus 3" is woolly because one word ("minus") is being used to describe two operations: the binary operation of subtraction ("take away") and also the unary operation of negation. Better, therefore, is to say "9 subtract negative 3" or "9 take away negative 3", or even "9 minus negative 3" or "9 subtract minus 3" – just avoid using the same word "minus" to describe different operations.