MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Multiplication and Division

2.24 Division: dividing by two-digit divisors

Teacher guide | Year 6

Teaching point 1:

Any two- or three-digit dividend can be divided by a two-digit divisor by skip counting in multiples of the divisor (quotient < 10); these calculations can be recorded using the short or long division algorithms.

Teaching point 2:

Any three- or four-digit dividend can be divided by a two-digit divisor using the short or long division algorithms (including quotient ≥ 10).

Teaching point 3:

When there is a remainder, the result can be expressed as a whole-number quotient and a whole-number remainder, as a whole-number quotient and a proper-fraction remainder, or as a decimal-fraction quotient.

Overview of learning

In this segment children will:

- carry out division calculations, that have quotients less than ten, using informal written methods, or short or long division, including:
  - *two- or three-digit ÷ two-digit* calculations where the divisor is a multiple of ten, both without remainders (e.g. *150 ÷ 30 = 5*) and with remainders (e.g. *85 ÷ 30 = 2 r 25*)
  - *two- or three-digit ÷ two-digit* calculations where the divisor is *not* a multiple of ten, again both without remainders (e.g. *93 ÷ 31 = 3*) and with remainders (e.g. *295 ÷ 32 = 9 r 7*)
- use their knowledge of doubling, halving and place value to efficiently create ratio charts to support calculation
- explore how partitioning in different ways (other than according to place value) supports calculation
- extend and compare use of the short and long division algorithms to carry out division calculations that have quotients greater or equal to ten, including:
  - *three-digit ÷ two-digit* calculations both without remainders (e.g. *434 ÷ 31 = 14*) and with remainders (e.g. *718 ÷ 33 = 21 r 25*)
  - *four-digit ÷ two-digit* calculations both without remainders (e.g. *4,945 ÷ 23 = 215*) and with remainders (e.g. *7,283 ÷ 28 = 260 r 3*)
- learn, when the dividend is not an exact multiple of the divisor, to express the answer as a quotient with a whole-number remainder, as a quotient with a proper-fraction remainder, or as a decimal fraction:

  $730 \div 25 = 29 \text{ r } 5$ $\quad$ $730 \div 25 = 29\frac{1}{5}$ $\quad$ $730 \div 25 = 29.2$

- evaluate which of the above representations of answers with remainders is appropriate based on a given context.

This segment introduces use of both the short and long division algorithms for division by two-digit divisors. Some key areas of focus are:

- for children to gain a deep understanding of the underlying mathematics involved (i.e. to understand *how* the algorithms work rather than just learning a method to apply by rote)
- the fact that the short and long multiplication algorithms are not mathematically different, but instead are two different ways to represent the same calculation process
- for children to be able to make sensible choices about which strategies and representations to use when presented with a calculation.

This segment builds on:

- multiplication of a single-digit number by a multiple of ten (segment *2.13 Calculation: multiplying and dividing by 10 or 100*)
- multiplication of any two-digit number by ten (segment *2.13*)
- using equivalence to divide one multiple of ten by another (segment *2.18 Using equivalence to calculate*)
- use of the short-division algorithm for calculations with single-digit divisors (segment *2.15 Division: partitioning leading to short division*)
- doubling and halving strategies (segment *2.5 Commutativity (part 2), doubling and halving*).

Children must already be fluent in these areas, as well as with all times-table facts, before they learn how to calculate with two-digit divisors.

As discussed, application of the short and long division algorithms to calculations with two-digit divisors is taught in a way that develops conceptual understanding of division rather than rote learning of the methods. This is especially important in the context of long division, where children must understand *why* digits are written in certain places in order to avoid place-value errors. As with other segments on formal written methods, the use of unitising language supports this understanding.

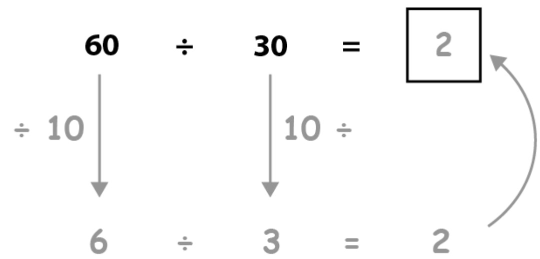
An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

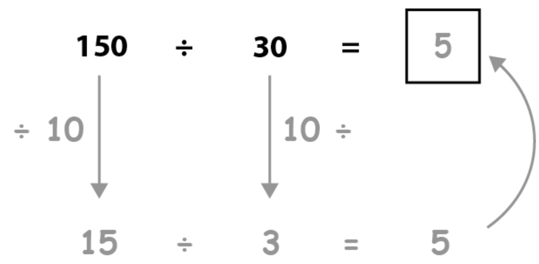
Teaching point 1:

Any two- or three-digit dividend can be divided by a two-digit divisor by skip counting in multiples of the divisor (quotient < 10); these calculations can be recorded using the short or long division algorithms.

Steps in learning

	Guidance	Representations

1:1	In this teaching point, children will use their knowledge of multiplication and skip counting to divide two- and three-digit dividends by two-digit divisors. They will be encouraged to use estimation skills and their understanding of place value.
	
	Steps *1:1–1:3* explore cases where the dividend is a multiple of ten. Children will use known multiplication facts, and their understanding of the effect on the product of making a factor ten times the size, to skip count in the divisor (see segment *2.13 Calculation: multiplying and dividing by 10 or 100*) or their understanding that if the dividend and divisor are each divided by the same value, the quotient remains the same (see segment *2.18 Using equivalence to calculate*). Steps *1:4–1:7* will explore cases where the dividend is *not* a multiple of ten. Children will use their estimation skills to work towards finding the quotient, based on their knowledge of dividing by multiples of ten. Throughout this teaching point, keep to examples where the quotient is less than ten.
	
	Begin by presenting a *two-digit ÷ two-digit* calculation where the dividend is a multiple of the divisor, and the divisor is a multiple of ten (e.g. *60 ÷ 30*). Discuss the following strategies for solving this problem:
	
	• skip counting in multiples of 30:
	    • based on skip counting in multiples of three, and using
		Two-digit ÷ multiple-of-ten calculation; no remainder:
		'Sixty children are placed into classes of thirty. How many classes are there?'
		
		60 ÷ 30 = ?
		
		• Skip counting in multiples of thirty
		    • *'Three tens, six tens.'*
		    • *'Thirty, sixty.'*
		    • **2** × 30 = 60
		    so
		    60 ÷ 30 = **2**
		
		• Scaling the dividend and divisor
		
		
		• Recording as short division
		      0 2
		30) 6 ^{6}0

	children's understanding from segment *2.13*: ***'If one factor is made ten times the size, the product will be ten times the size.'*** or • unitising language. • using the known single-digit fact and children's understanding, from segment *2.18*, step *2:4*: ***'If I divide the dividend by ten, I must divide the divisor by ten for the quotient to stay the same.'*** Illustrate how this can be recorded using the short-division algorithm. Work through a few other examples in the same way, including three-digit dividends, but keeping to quotients less than ten and no remainders, for example: • 80 ÷ 20 • 150 ÷ 30 • 360 ÷ 90	*Three-digit ÷ multiple-of-ten* calculation; no remainder: 150 ÷ 30 = ? • Skip counting or unitising **5** × 3 = 15 so **5** × 3 tens = 15 tens **5** × 30 = 150 150 ÷ 30 = **5** • Scaling the dividend and divisor  • Recording as short division 0 0 5 30) 1 [1]5 [15]0
1:2	Now work through a problem with a remainder, still keeping to a multiple-of-ten divisor and a quotient less than ten (e.g. *85 ÷ 30*). First estimate the answer, as illustrated opposite, and use informal written methods to complete the calculation. Then illustrate how this can be recorded using the long-division algorithm. Since this is the first use of the long-division algorithm, take some time to explain the structure, using the example problem, and emphasise the proper alignment of the digits. Use similar language to that described in segment *2.15 Division: partitioning leading to short division*:	*Two-digit ÷ multiple-of-ten* calculation; with remainder: *'Eighty-five conkers are shared equally between thirty children. How many conkers does each child get?'* 85 ÷ 30 = ? • Estimation • 2 × 30 < 85 < 3 × 30 so 2 < 85 ÷ 30 < 3 • *'The answer is between two and three.'*

- *'First write the divisor (30), the frame, and the dividend (85).'*
- *'Now divide: the largest multiple of thirty less than eighty-five is sixty; write "60" beneath "85", aligning the digits…*
- *'…sixty divided by thirty is equal to two; write "2" in the ones column of the answer line.'*
- *'Now calculate the remainder: eighty-five minus sixty is equal to twenty-five; write "r 25" at the end of the answer line.'*

Ask questions to ensure that children can explain what each digit (and number) represents in the algorithm; for example:

- *'What does the "85" represent?'*
- *'What does the "30" represent?'*
- *'What does the "60" represent?'*
- *'What does the "2" represent?'*
- *'What does the "25" represent?'*

Work through some other examples with remainders, including three-digit dividends, but keeping to quotients less than ten and multiple-of-ten divisors, for example:

- $72 \div 20$
- $87 \div 30$
- $345 \div 90$
- $361 \div 40$
- $519 \div 70$

- Informal written method

  $60 \div 30 = 2$ (from step *1:1*)

  $85 - 60 = 25$

so

  $85 \div 30 = 2 \text{ r } 25$

- Recording as long division

```
        2  r 25
    ___________
30 ) 8  5
     6  0
     2  5
```

- *'Each child gets two conkers. There are twenty-five conkers left over.'*

Three-digit ÷ multiple-of-ten calculation; with remainder:

  $364 \div 50 = ?$

- Estimation
  - $7 \times 50 < 364 < 8 \times 50$

  so

  $7 < 364 \div 50 < 8$
  - *'The answer is between seven and eight.'*

- Informal written method

  $7 \times 50 = 350$

  $350 \div 50 = 7$

  $364 - 350 = 14$

so

  $364 \div 50 = 7 \text{ r } 14$

- Recording as long division

```
           7  r 14
    ______________
50 ) 3  6  4
     3  5  0
     0  1  4
```

<table>
<tr>
<td>1:3</td>
<td>Provide children with some practice laying out and completing two-digit ÷ two-digit and three-digit ÷ two-digit calculations, with multiple-of-ten divisors and quotients less than ten. Children should lay out and perform the calculations using the division algorithm (short or long as appropriate), but encourage them to precede each calculation with estimation and then use this to sense-check their answers.</td>
<td>'Complete the calculations.'
60 ÷ 20   540 ÷ 60

20) 6 3   60) 5 5 5

63 ÷ 20   97 ÷ 40   93 ÷ 70</td>
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<tr>
<td>1:4</td>
<td>Now progress to calculations where the divisor is not a multiple of ten, but for which there is no remainder, keeping to quotients less than ten.

Begin with a two-digit ÷ two-digit calculation (e.g. 93 ÷ 31). Note that this first example has a divisor that is easily multiplied by the required number without the ones crossing a tens boundary.

Use estimation to get a trial value for the quotient, as shown opposite, and record using long division. You can use place-value counters for support (note that children learnt to use partitioning to multiply a two-digit number by a single-digit number in segment 2.14 Multiplication: partitioning leading to short multiplication, Teaching point 1).

Ask questions to ensure that children can explain what each digit (and number) represents in the algorithm; for example:
• 'What does the "93" represent?'
• 'What does the "31" represent?'
• 'Which number is the divisor?'
• 'Which number is the dividend?'
• 'Which number is the quotient?'
• 'What does the quotient represent?'

Work through a few other examples in the same way, including three-digit dividends, but keeping to quotients</td>
<td>Two-digit ÷ two-digit calculation; no remainder:
'There are <u>ninety-three</u> school newsletters. Each class needs <u>thirty-one</u> newsletters. There are enough newsletters for how many classes?'
93 ÷ 31 = ?

• Estimate
31 is close to 30.
  • 'Roughly how many 'thirties' are there in ninety-three?'
  3 × 3 tens = 9 tens
  3 × 30 = 90
  • 'There are roughly three 'thirties' in ninety-three.'

• So, try subtracting 3 × 31 from 93
      3
31) 9 3
    9 3   (3 × 31)
    0 0
so
93 ÷ 31 = 3</td>
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<td></td>
<td>less than ten and no remainders, for example:
• 88 ÷ 22
• 408 ÷ 51
• 276 ÷ 92
For now, use divisors that are always rounded down for the estimation step (i.e. ones-digit < 5).</td>
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<td>1:5</td>
<td>Now work through a problem where there is a remainder, still keeping to a quotient less than ten, and a divisor with a ones digit less than five (e.g. 295 ÷ 32).</td>
<td>Two-digit ÷ two-digit calculation; with remainder:
295 ÷ 32 = ?

• Estimate:
32 is close to 30.
• 'Roughly how many 'thirties' are there in two hundred and ninety-five?'
9 × 3 tens = 27 tens
9 × 30 = 270
• 'There are roughly nine 'thirties' in two hundred and ninety-five.'

• So, try subtracting 9 × 32 from 295
          9 r 7
32)2 9 5
     2 8 8   (9 × 32)
     0 0 7
so
295 ÷ 32 = 9 r 7</td>
</tr>
<tr>
<td>1:6</td>
<td>Now show what happens when using the estimation results in a remainder that is larger than the divisor. Children already know that the remainder must be less than the divisor, so discuss and demonstrate how this means we must increase the quotient by one, as shown opposite and on the next page.</td>
<td>Two-digit ÷ two-digit calculation; with remainder (using the original estimate gives remainder > divisor):
297 ÷ 37 = ?

• Estimate
37 is close to 40.
• 'Roughly how many 'forties' are there in two hundred and ninety-seven?'
7 × 4 tens = 28 tens
7 × 40 = 280
• 'There are roughly seven forties in two hundred and ninety-seven.'</td>
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</table>

		• So, try subtracting 7×37 from 297 $\begin{array}{r} 7 \\ 37\overline{)2\ 9\ 7} \\ \underline{2\ 5\ 9} \\ 0\ 3\ 8 \end{array} \quad (7 \times 37)$ • *'The remainder cannot be greater than the divisor.'* • So, we can subtract **8** × 37 from 297: $\begin{array}{r} 8\ \ r\,1 \\ 37\overline{)2\ 9\ 7\ \ \ \ } \\ \underline{2\ 9\ 6} \\ 0\ 0\ 1 \end{array} \quad (8 \times 37)$ so $297 \div 37 = 8 \text{ r } 1$

1:7	Give children practice laying out and completing *two-digit* ÷ *two-digit* and *three-digit* ÷ *two-digit* calculations where the divisor is not a multiple of ten, including:
	• abstract problems
	• error-spotting, and correcting calculations
	• contextual problems, including quotitive and partitive structures (problems with the scaling structure are considered in *Teaching point 3*):
	  • *'364 eggs are shared equally between 52 boxes. How many eggs are there in each box?'* (partitive division)
	  • *'There are 24 hours in a full day. How many full days are there in 205 hours?'* (quotitive division)
	Include examples both with and without remainders.
	Encourage children to gradually simplify their working for cases without a remainder, i.e.:
		Abstract problems:
		'Complete the calculations.'
		$91 \div 13$ $74 \div 28$ $91 \div 28$
		$97 \div 42$ $82 \div 14$ $90 \div 18$
		$102 \div 17$ $125 \div 16$ $108 \div 12$
		$543 \div 52$ $132 \div 22$ $530 \div 75$
		Error-spotting:
		'Spot the mistakes. Explain your answers and correct the calculations.'
		$\begin{array}{r} 2 \\ 28\overline{)8\ 4} \\ \underline{5\ 6} \\ 2\ 8 \end{array} \quad \begin{array}{r} 9 \\ 52\overline{)4\ 3\ 6} \\ \underline{4\ 6\ 8} \\ 3\ 2 \end{array} \quad \begin{array}{r} 8 \\ 55\overline{)4\ 5\ 0} \\ \underline{4\ 5\ 0} \\ 0 \end{array}$
		Dòng nǎo jīn:
		'Fill in the missing digits in the boxes.'
		$\begin{array}{r} ?\ \ ?\ \ ?\ \ r\,9 \\ 30\overline{)\square\ \square\ \square\ \ \ \ \ } \end{array}$
		'How many solutions can you find?'

	$\begin{array}{r} 9 \\ 12\overline{)1\;0\;8} \\ 1\;0\;8 \\ \hline 0 \end{array}$ *'If the dividend is a multiple of the divisor, I can record it like this:'* $\begin{array}{r} 9 \\ 12\overline{)1\;0\;8} \end{array}$	

Teaching point 2:

Any three- or four-digit dividend can be divided by a two-digit divisor using the short or long division algorithms (including quotient ≥ 10).

Steps in learning

	Guidance	Representations
2:1	In this teaching point, the short and long division algorithms are both used and compared. First, spend some time reviewing multiplication strategies (such as doubling and halving) for calculating multiples of the divisor. Children already know how to multiply any two-digit number by a single-digit number using partitioning, but focus on quick and efficient methods based around doubling, halving and place-value; use a table to record the multiples clearly. Begin with a number such as 21 (this number has been chosen as the first example because it is relatively easy to double a few times). Work through filling in as much of the multiples table as possible, based on doubling (*2 × 21*, *4 × 21* and *8 × 21*), asking children to explain how each multiple is calculated. Then calculate the tenth multiple (*10 × 21*; see segment *2.13 Calculation: multiplying and dividing by 10 or 100*), and after that ask children if their halving skills could give them another multiple (*5 × 21*). Practise with other starting numbers e.g. 32, 45, 37 and 59.	(see table below)
2:2	Now present a *three-digit ÷ two-digit* calculation, with a quotient greater than ten and with no remainder; for example, *'Becky has 434 cm of ribbon to wrap up prizes for a school competition. Each prize needs 31 cm of ribbon. How many prizes can she wrap?'* Work to build a ratio chart for multiples	

	× 21
1	21
2	42
3	
4	84
5	105
6	
7	
8	168
9	
10	210

of the divisor (31) as described in step *2:1*. Then return to the calculation (*434 ÷ 31*) and, reminding children of how they used partitioning to calculate in segment *2.15 Division: partitioning leading to short division*, explore how the dividend (434) can be partitioned to help us carry out the calculation. Most children will first partition into hundreds, tens and ones (*400 + 30 + 4*). Discuss how none of these parts are divisible by the divisor (31). Support children to partition into multiples of 31 instead (using the ratio chart), as shown opposite.

Ratio chart and partitioning:

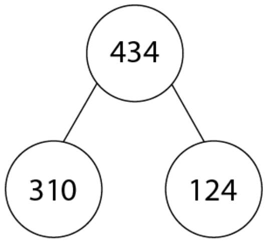
'Becky has 434 cm of ribbon to wrap up prizes for a school competition. Each prize needs 31 cm of ribbon. How many prizes can she wrap?'

434 ÷ 31 = ?

- Create a ratio chart of multiples of the divisor

	× 31
1	31
2	62
3	
4	124
5	155
6	
7	
8	248
9	
10	310

- Partition the dividend to calculate



310	÷	31	=	10
124	÷	31	=	4
434	÷	31	=	14

- *'Becky can wrap fourteen presents.'*

2:3

Now use the short division algorithm to carry out the calculation, using the ratio chart for support. In the same way as in segment *2.15 Division: partitioning leading to short division*, describe the steps in full as you work through the calculation:

- *'First write the divisor (31), the frame, and the dividend (434).'*
- *'Now divide, starting with the hundreds: four hundreds divided by thirty-one is equal to zero hundreds, with a remainder of four hundreds; write "0" in the hundreds column…'*
- *'…and exchange the remainder: four hundreds is equal to forty tens: write "4" to the left of the tens digit of the dividend to make forty-three tens.'*
- *'Then move to the tens: forty-three tens divided by thirty-one is equal to one ten, with a remainder of twelve tens; write "1" in the tens column…'*
- *'…and exchange the remainder: twelve tens is equal to one hundred and twenty ones: write "12" to the left of the ones digit of the dividend to make one hundred and twenty-four ones.'*
- *'Then move to the ones: one hundred and twenty-four ones divided by thirty-one is equal to four ones; write "4" in the ones column.'*

Ratio chart and short division:

'Becky has 434 cm of ribbon to wrap up prizes for a school competition. Each prize needs 31 cm of ribbon. How many prizes can she wrap?'

434 ÷ 31 = ?

	× 31
1	31
2	62
3	
4	124
5	155
6	
7	
8	248
9	
10	310

0 31)4 3 4	4 hundreds ÷ 31 = 0 hundreds r 4 hundreds • *'Write "0" in the hundreds column…'*
0 31)4 ^{4}3 4	4 hundreds = 40 tens • *'…and write "4" to the left of the tens digit of the dividend.'*
0 1 31)4 ^{4}3 4	43 tens ÷ 31 = 1 ten r 12 tens • *'Write "1" in the tens column...'*
0 1 31)4 ^{4}3 ^{12}4	12 tens = 120 ones • *'…and write "12" to the left of the ones digit of the dividend.'*
0 1 4 31)4 ^{4}3 ^{12}4	124 ones ÷ 31 = 4 ones (refer to the ratio chart) • *'Write "4" in the ones column.'*

- *'Becky can wrap fourteen presents.'*

2:4

Now carry out the same calculation recording using the long-division algorithm, again using the ratio chart for support. Use similar descriptive sentences to those for short division in step *2:3*, but amend to include recording the calculation of the remainders at each stage. Note that the final subtraction step (*124 – 124*) can be omitted in due course, but is included at this stage for completeness.

Throughout, pay particular attention to the value of the digits and the importance of writing each digit in the correct column.

Ratio chart and long division:

'Becky has 434 cm of ribbon to wrap up prizes for a school competition. Each prize needs 31 cm of ribbon. How many prizes can she wrap?'

434 ÷ 31 = ?

Ratio chart:

	× 31
1	31
2	62
3	
4	124
5	155
6	
7	
8	248
9	
10	310

Step 1 – write the divisor, frame and dividend

31)4 3 4

Step 2 – divide the hundreds

  0
31)4 3 4

4 hundreds ÷ 31 = 0 hundreds r 4 hundreds

- *'Write "0" in the hundreds column of the answer line.'*

Step 3 – exchange hundreds for tens, combine with the existing tens and divide…

  0 1
31)4 3 4
  3 1   (1 ten × 31 = 31 tens)

4 hundreds = 40 tens

40 tens + 3 tens = 43 tens

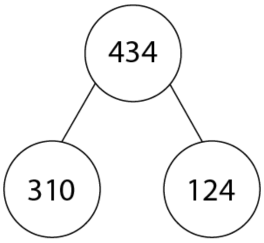
43 tens ÷ 31 = 1 ten and a remainder

- *'Write "1" in the tens column of the answer line and write "31" underneath the "43".'*

	Step 4 – subtract to find the remainder
	`    0  1`
	`31)4  3  4`
	`   3  1      (1 ten × 31 = 31 tens)`
	`   1  2`
	43 tens – 31 tens = 12 tens
	• *'Write "12" underneath the "31".'*
		Step 5 – exchange tens for ones and combine with the existing ones
		`    0  1`
		`31)4  3  4`
		`   3  1  ↓     (1 ten × 31 = 31 tens)`
		`   1  2  4`
		12 tens = 120 ones
		120 ones + 4 ones = 124 ones
		• *'Write "4" after the "12".'*

	Step 6 – divide the ones
	`    0  1  4`
	`31)4  3  4`
	`   3  1         (1 ten × 31 = 31 tens)`
	`   1  2  4`
	`   1  2  4      (4 ones × 31 = 124 ones)`
	124 ones ÷ 31 = 4 ones
	(refer to the ratio chart)
	• *'Write "4" in the ones column of the answer line and write "124" underneath the "124", aligning the digits.'*
		Step 7 – subtract to show there is no remainder
		`    0  1  4`
		`31)4  3  4`
		`   3  1         (1 ten × 31 = 31 tens)`
		`   1  2  4`
		`   1  2  4      (4 ones × 31 = 124 ones)`
		`         0`
		124 ones – 124 ones = 0 ones
		• *'Write "0" underneath the "31".'*

	• *'Becky can wrap fourteen presents.'*	
2:5	Now compare the two algorithms and the partitioning method. Ask children: • *'What's the same?'* • *'What's different?'* Draw attention to the fact that the underlying mathematics is the same in both algorithms but the long-division algorithm allows us to show calculation of the remainders. Discuss how both algorithms connect to partitioning the dividend into multiples of the divisor (and how this can be seen more easily in the long division calculation).	

Partitioning	Short division	Long division
 310 ÷ 31 = 10 124 ÷ 31 = 4 434 ÷ 31 = 14	0 1 4 31)4 ^{4}3 ^{12}4	0 1 4 31)4 3 4 3 1 (1 ten × 31 = 31 tens) 1 2 4 1 2 4 (4 ones × 31 = 124 ones) 0

2:6 Work through some other *three-digit* ÷ *two-digit* calculations. For each, first create a ratio chart, then work through both the short- and long-division algorithms. Include calculations:

- with no remainder (e.g. *483 ÷ 21*)
- with a remainder (e.g. 718 ÷ 33), expressing the remainder as a whole number; alternative representations of remainders will be explored in *Teaching point 3*.

Note that there is often more than one way to efficiently partition the dividend when working informally.

Three-digit ÷ two-digit calculation with a remainder:

Ratio chart

	× 33
1	33
2	66
3	
4	132

Short division

0 2 1 r 25
33)7 ^{7}1 ^{5}8

Long division

2 1 r 25
33)7 1 8
6 6
5 8
3 3
2 5

2:7 At this point, provide children with practice completing *three-digit ÷ two-digit* calculations, where the quotient is greater or equal to ten, both with and without remainders. Include both abstract and contextual problems.

It is worth also including some problems with the quotient less than ten (as in *Teaching point 1*), to ensure that children can decide which method

'Complete the calculations.'

41)9 7 2   35)9 8 5

126 ÷ 21   739 ÷ 32

to use on a case-by-case basis (i.e. for the calculations *126 ÷ 21* and *224 ÷ 32*, children should recognise that 12 is smaller than 21, and 22 smaller than 32, and then revert to the methods used in *Teaching point 1*).

Example word problems:

- *'224 stickers are shared equally between 32 children. How many stickers does each child get?'* (partitive division)
- *'A school has 652 pupils and 40 members of staff. If a bus can transport 52 people, how many buses are needed for the whole school to visit the seaside?'* (addition and quotitive division)

Dòng nǎo jīn:

'Fill in the missing multiples.'

1	
2	
3	249
4	
5	
6	498
7	
8	
9	
10	

'Now write a division calculation with a three-digit dividend and a remainder of five, using the two-digit divisor from the top row of the table.'

2:8

Now extend to *four-digit ÷ two-digit* calculations, first without remainders (e.g. *4,945 ÷ 23*), then with remainders (e.g. *7,283 ÷ 28*). Continue to use ratio charts for support, and encourage children to use unitising language. Note that the example with a remainder opposite has a quotient with a ones digit of zero; a common error is for children to omit the zero in the ones place.

Children can use either the short or the long division algorithm for any of the calculations; encourage them to find which method works best for them.

Four-digit ÷ two-digit calculation without remainder:

4,945 ÷ 23 = ?

Long division

```
      2  1  5
23)4  9  4  5
   4  6
      3  4
      2  3
      1  1  5
      1  1  5
            0
```

Short division

```
    0   2   1   5
23) 4  ⁴9  ³4 ¹¹5
```

Four-digit ÷ two-digit calculation with remainder:

7,283 ÷ 28 = ?

Long division

```
      2  6  0  r3
28)7  2  8  3
   5  6
   1  6  8
   1  6  8
         0  3
```

Short division

```
                   3
28) 7  ⁷2  ¹⁶8   3
```


2:9

To complete this teaching point, provide children with general practice for both *three-digit ÷ two-digit* and *four-digit ÷ two-digit* calculations, with and without remainders in the form of:

- completing calculations (ensure you provide a range of calculations including those with a zero in either the tens place or ones place of the quotient, and *four-digit ÷ two-digit* calculations with a two-digit quotient)
- reasoning problems, such as those shown opposite
- missing-digit problems, to deepen children's understanding
- error-spotting and correcting calculations
- contextual problems, including both the partitive and quotitive structures of division, for example:
  - *'Seven thousand marbles are put into packs of fifty-six. How many packs are made?'* (quotitive division)
  - *'A school is given £2350 to buy new reading books. The money is shared equally between the twelve classes. Any left-over money will be spent by the headteacher.'*
    - *'How much money does each class get?'*
    - *'Can the headteacher buy a book that costs £10.99?'*

    (partitive division)

Completing calculations:

'Complete the calculations.'

42)8 6 1

42)8 6 1 0

1,887 ÷ 18

1,887 ÷ 37

Reasoning and missing-digit problems:

- *'What could the missing digits be if this calculation has a whole number quotient?'*

  23)1 ☐ ☐ 5

  'How many solutions can you find?'

- *'The diamonds (◊) both represent the same digit. What digit could the diamonds represent if this calculation has a quotient with a remainder of three?'*

  r 3

  16)8 ◊ ◊

- *'Fill in the missing digit.'*

  ☐92 ÷ 14 = 28

Error-spotting:

'Spot the mistakes. Explain your answers and correct the calculations.'

```
        3 4
26)7 9 0 4
   7 8 ↓ ↓
     1 0 4
     1 0 4
         0
```

```
      2   r 28
36)7 4 8
   7 2 ↓
     2 8
```

```
        7 0
46)3 5 3
   3 2 2
     3 1
```

Teaching point 3:

When there is a remainder, the result can be expressed as a whole-number quotient and a whole-number remainder, as a whole-number quotient and a proper-fraction remainder, or as a decimal-fraction quotient.

Steps in learning

	Guidance	Representations
3:1	see below	see below

3:1 — Guidance

Now that children can confidently divide by two-digit divisors, explore the different ways to express the outcome of calculations with remainders, including:

- as a whole-number quotient and a whole-number remainder (e.g. *730 ÷ 25 = 29 r 5*)
- as a decimal fraction (e.g. *730 ÷ 25 = 29.2*)
- as a whole-number quotient and a proper-fraction $\left(\text{e.g. } 730 \div 25 = 29\frac{1}{5}\right)$.

When introducing the three options, the dividend and divisor should be kept the same, as exemplified opposite, so that the focus is on the difference in the way the remainder is expressed. Note that, in steps *3:1–3:4*, calculations are selected such that when the answer is expressed as a decimal fraction, only one decimal place is required.

Before beginning, you may wish to recap learning from segment *2.12 Division with remainders*, *Teaching point 3* regarding the careful interpretation of calculations with remainders in connection with contextual problems. Then apply this to a problem solved using long division as illustrated opposite (children already learnt to express remainders as, for example, *29 r 5* in *Teaching point 2*).

3:1 — Representations

Long division – remainder expressed as a whole number:

'Sue has seven hundred and thirty books. She packs them into boxes of twenty-five.'

- *'How many full boxes are there?'*
- *'How many boxes does she need to pack all of the books?'*
- *'How many books are not in a full box?'*

730 ÷ 25 = ?

```
       2  9  r 5
   25)7  3  0
      5  0
      2  3  0
      2  2  5
      0  0  5
```

	× 25
1	25
2	50
4	100
5	125
8	200
10	250

- So, 730 ÷ 25 = 29 r 5

Full boxes: 29

Boxes needed: 30

Books not in a full box: 5

Short division alternative:

$$\begin{array}{r} 0\ \ \ 2\ \ \ 9\ \ \text{r}\,5 \\ 25\overline{)\,7\ \ {}^{7}3\ \ {}^{23}0} \end{array}$$

3:2	Now present a context with the same dividend and divisor as in step *3:1*, but for which it would be more appropriate to express the result as a decimal fraction; for example: '*Dinesh sells twenty-five jumpers, each for the same amount, and makes a total of £730. How much did each jumper sell for?*'
	
	Discuss why it isn't appropriate to use the answer from step *3:1*, because it only tells us that each jumper costs £29 'and a bit' and we need to know the exact price. Note that, in this particular case, expressing the remainder as a fraction, as in the next step, could be used to convert the answer, if children know that one-fifth of £1 is 20 p.
	
	Demonstrate how to continue the application of the long division algorithm to work out the value of the digit after the decimal point. Draw attention to the alignment of the decimal point, and exchange of 5 ones for 50 tenths. As usual, describe the whole-number part of the calculation with unitising until you reach the remainder (*Step 1* opposite); then continue the calculation, describing the introduction of the decimal point (*Step 2* opposite) and unitising in tenths (*Steps 3* and *4* on the next page).
	
	A common error is for children to write the remainder in the tenths column (resulting in an incorrect answer of 29.5 in this example), without taking into account that the remainder (5) needs to be divided by the divisor (25).
		Long division – remainder converted to decimal fraction:
		
		'Dinesh sells twenty-five jumpers, each for the same amount, and makes a total of £730. How much did each jumper sell for?'
		
		730 ÷ 25 = ?
		
		Step 1 – calculate the whole-number quotient:
		
		      2 9
		25)7 3 0
		    5 0
		    2 3 0
		    2 2 5
		        5
		
		• 73 tens ÷ 25 = 2 tens and a remainder
		'Write "2" in the tens column of the answer line and write "50" underneath the "73".'
		• 73 tens – 50 tens = 23 tens
		'Write "23" underneath the "50".'
		• 23 tens = 230 ones
		'Write "0" after the "23".'
		• 230 ones ÷ 25 = 9 ones and a remainder
		'Write "9" in the ones column of the answer line and write "225" underneath the "230".'
		• 230 ones – 225 ones = 5 ones
		'Write "5" underneath the "225".'
		
		Step 2 – introduce the decimal point:
		
		      2 9 .
		25)7 3 0 . 0
		    5 0
		    2 3 0
		    2 2 5
		        5
		
		• *'There is a remainder. To represent this as a decimal fraction, first write a decimal point after the ones digit of both the dividend and the quotient. Write a place-holder zero in the tenths column of the dividend.'*

Step 3 – continue:

```
          2   9 .
    ___________
25 ) 7    3   0 . 0
     5    0
     -----
     2    3   0
     2    2   5
     ----------
              5   0
```

- 5 ones = 50 tenths
  'Write "0" after the "5".

Step 4 – complete the calculation:

```
          2   9 . 2
    _______________
25 ) 7    3   0 . 0
     5    0
     -----
     2    3   0
     2    2   5
     ----------
              5   0
              5   0
              -----
                  0
```

- 50 tenths ÷ 25 = 2 tenths
  'Write "2" in the tenths column of the answer line and write "50" underneath the "50".'
- 50 tenths – 50 tenths = 0 tenths
  'Write "0" underneath the "50".'

- So, 730 ÷ 25 = 29.2
- *'Each jumper sold for £29.20.'*

Short division alternative:

```
     0    2     9  .  2
    ___________________
25 ) 7   ⁷3   ²³0  . ⁵0
```

3:3	Keeping the same dividend and divisor, explore a context for which it would be appropriate to express the result as a proper fraction. Again, use the usual descriptive language to explain the calculation until you reach the remainder; then describe as exemplified on the next page.	

	When providing the final answer to the question, it is important for children to express the fractional remainder in its simplest form; children learnt to simplify fractions in *Spine 3: Fractions*, segment *3.7*. Note that in the example opposite, it would also be appropriate to say that Saveeta travels 29.2 times the distance that Lily travels.	Long division – remainder converted to proper fraction: *'Saveeta travels 730 km to visit her grandma. Lily travels 25 km to visit her grandma. How many times Lily's distance does Saveeta travel?'* $730 \div 25 = ?$ $\begin{array}{r} 2\;9\;\frac{5}{25} \\ 25\overline{)7\;3\;0} \\ \underline{5\;0} \\ 2\;3\;0 \\ \underline{2\;2\;5} \\ 5 \end{array}$ • *'There is a remainder. To represent this as a fraction, write the remainder as the numerator and the divisor as the denominator.'* $5 \div 25 = \frac{5}{25}$ *'Write* $\frac{5}{25}$ *after the ones column of the answer line.'* • *'Simplify the fraction.'* $\frac{5}{25} = \frac{1}{5}$ • So, $730 \div 25 = 29\frac{1}{5}$ • *'Saveeta travels twenty-nine and one-fifth times the distance that Lily travels.'* Short division alternative: $\begin{array}{r} 0\;\;2\;\;9\;\frac{5}{25} \\ 25\overline{)7\;\;{}^{7}3\;\;{}^{23}0} \end{array}$
3:4	Using a different calculation, show three algorithms alongside each other, representing the three different ways of expressing the remainder. Then discuss, for a variety of contexts, which representation is appropriate. Note that there isn't necessarily one 'correct' answer; in many cases both the decimal fraction and the proper fraction representation of the remainder are appropriate.	

354 ÷ 15 = ?

$$\begin{array}{r}
23 \text{ r } 9 \\
15 \overline{)\,354} \\
\underline{30} \\
54 \\
\underline{45} \\
9
\end{array}
\qquad
\begin{array}{r}
23\tfrac{9}{15} \\
15 \overline{)\,354} \\
\underline{30} \\
54 \\
\underline{45} \\
9
\end{array}
\qquad
\begin{array}{r}
23.6 \\
15 \overline{)\,354.0} \\
\underline{30} \\
54 \\
\underline{45} \\
90 \\
\underline{90} \\
0
\end{array}$$

So, $354 \div 15 = 23 \text{ r } 9$	$\frac{9}{15} = \frac{3}{5}$ So, $354 \div 15 = 23\frac{3}{5}$	So, $354 \div 15 = 23.6$

Possible contexts:

- *'Robert raised £354 for charity. Harrison raised £15. How many times as much did Robert raise compared to Harrison?'*
- *'Debo has cut 354 m of rope into fifteen equal lengths for climbers to use. How long is each piece?'*
- *'Rachel has 354 beads. She uses fifteen to make a bracelet.'*
  - *'How many bracelets can she make?'*
  - *'How many beads does she have left over?'*
- *'Some children wash cars to earn some money for their holidays. There are fifteen children and they earn £354 altogether. How much does each child get if they share the money equally?'*
- *'One bin holds 15 kg of rubbish. During a school week 354 kg of rubbish is collected. How many bins have been filled?'*

3:5 Now that the children have an efficient method to divide, work through a calculation in which the quotient has two decimal places, beginning with cases that have '25' or '75' after the decimal point of the quotient (e.g. *105 ÷ 20 = 5.25*). Use similar language to describe the stages of the calculation as that exemplified in steps *3:2* and *3:3*.

Then work through some examples that have answers with two decimal places but that don't result in '.25' or '.75', e.g. *141 ÷ 60* and *236 ÷ 50*.

$105 \div 20 = ?$

$$\begin{array}{r} 5\,\tfrac{5}{20} \\ 20\overline{)1\;0\;5} \\ \underline{1\;0\;0} \\ 5 \end{array}$$

- *'There is a remainder. To represent this as a fraction, write the remainder as the numerator and the divisor as the denominator.'*
- $5 \div 20 = \frac{5}{20}$

  'Write "$\frac{5}{20}$" after the ones column of the answer line.'
- *'Simplify the fraction.'*

  $\frac{5}{20} = \frac{1}{4}$
- So, $105 \div 20 = 5\frac{1}{4}$

$$\begin{array}{r} 5\,.\,2\;5 \\ 20\overline{)1\;0\;5\,.\,0\;0} \\ \underline{1\;0\;0} \\ 5\;0 \\ \underline{4\;0} \\ 1\;0\;0 \\ \underline{1\;0\;0} \\ 0 \end{array}$$

- *'There is a remainder. To represent this as a decimal fraction, first write a decimal point after the ones digit of both the dividend and the quotient.'*
- *'Write a place-holder zero in the tenths column of the dividend.'*
- 5 ones = 50 tenths

  'Write "0" after the "5".'
- 50 tenths ÷ 20 = 2 tenths and a remainder

  'Write "2" in the tenths column of the answer line and write "40" underneath the "50".'
- 50 tenths − 40 tenths = 10 tenths

  'Write "10" underneath the "50".'
- *'Write a place-holder zero in the hundredths column of the dividend.'*
- 10 tenths = 100 hundredths

  'Write "0" after the "10".'
- 100 hundredths ÷ 20 = 5 hundredths

  'Write "5" in the hundredths column of the answer line and write "100" underneath the "100".'
- 100 hundredths − 100 hundredths = 0 hundredths

  'Write "0" underneath the "100".'
- So, 105 ÷ 20 = 5.25

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Dòng nǎo jīn:

- *'Explain why all these calculations have a "2" in the tenths place and a "5" in the hundredths place.'*

  205 ÷ 20   305 ÷ 20   405 ÷ 20   505 ÷ 20

- *'Complete this calculation using long division. Give your answer as a decimal fraction.'*
  115 ÷ 20
  - *'Use your answer to predict what digits will appear in the tenths and hundredths places of the answers to these calculations.'*

    215 ÷ 20   315 ÷ 20   415 ÷ 20

    'Without using long or short division, work out the answers to the calculations.'

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<td>3:6</td>
<td>Briefly explore an example where the decimal fraction would go into many places; children should realise that if this happens, it is more appropriate to express the remainder as a proper fraction. For contexts where either the decimal-fraction quotient, or quotient and proper-fraction remainder are both reasonable, children should choose the proper fraction if the decimal fraction would go beyond a few decimal places.</td>
<td>

770 ÷ 30 = ?

$$\begin{array}{r} 25.66\ldots \\ 30\overline{)770.00\ldots} \\ \underline{60} \\ 170 \\ \underline{150} \\ 20\,0 \\ \underline{18\,0} \\ 2\,00 \\ \underline{1\,80} \\ 20 \\ \vdots \end{array}$$

$$\begin{array}{r} 25\tfrac{20}{30} \\ 30\overline{)770} \\ \underline{60} \\ 170 \\ \underline{150} \\ 20 \end{array}$$

$\frac{20}{30} = \frac{2}{3}$

so

$770 \div 30 = 25\frac{2}{3}$

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<td>3:7</td>
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To complete this teaching point, provide children with practice completing division calculations that involve remainders, including both abstract and contextual problems; for the latter, encourage children to think carefully about the best way to interpret and represent the remainder.

Include problems that have answers with:

- one decimal place
- '25' or '75' after the decimal point
- two decimal places (other than '.25' and '.75').

</td>
<td>

'Complete the calculations. For each calculation, show the remainder in three different ways.'

525 ÷ 42   930 ÷ 40

1026 ÷ 24   307 ÷ 25

</td>
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Example word problems:

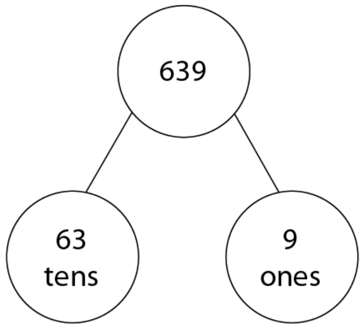
- *'There are 543 g of raisins. They are packed into snack-boxes each weighing 52 g. How many snack-boxes can be made?'*
(quotitive division)
- *'£1431 is shared equally between fifty people. How much does each person get?'*
(partitive division)
- *'A large paddling pool holds 530 litres of water. Exactly how many 15-litre buckets are needed to fill it?'*
(quotitive)
- *'Azra takes a 14 km train journey to school each day. This weekend, Azra will visit her grandma, taking a 175 km train journey. How many times the distance of her trip to school is this?'*
(scaling)

You could also provide a variety of completed calculations and ask children to write their own contexts appropriate to the way the remainder is expressed.

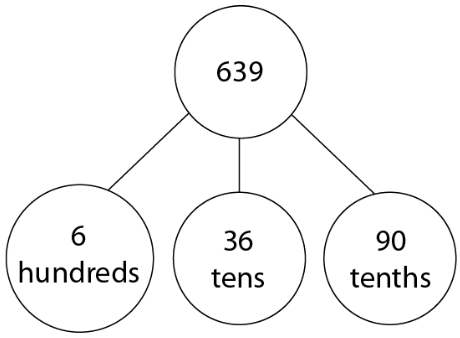
Dòng nǎo jīn:

- *'Sammy calculates 639 ÷ 12 using long division. She realises that this is the same as partitioning 639, but not as 6 hundreds, 3 tens and 9 ones. Which way has 639 been partitioned?'*

A



B



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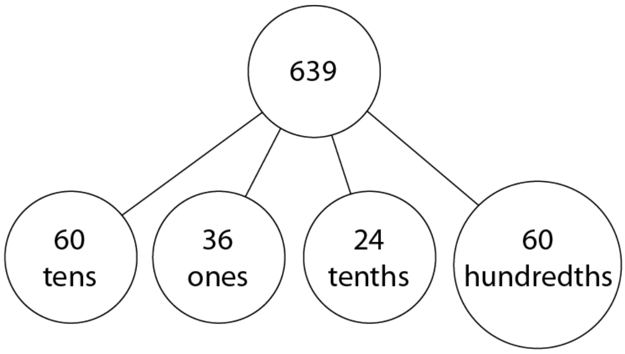
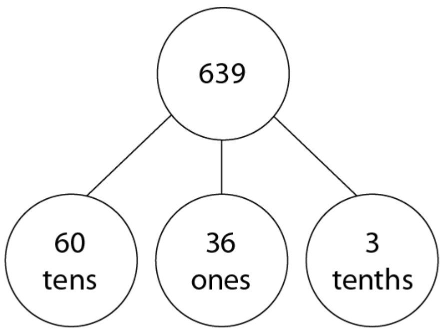


D



		• *'Fill in the missing digits in the boxes.'* ? ? . 7 5 24)□ □ □