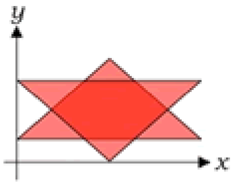
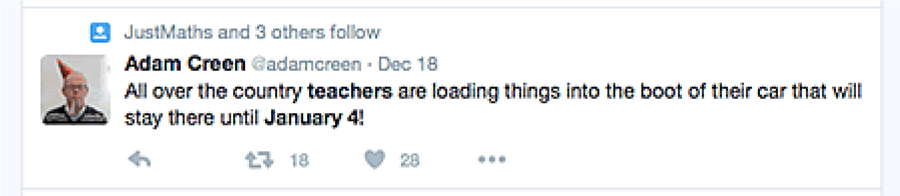
Secondary Magazine 129



National Centre for Excellence in the Teaching of Mathematics

Welcome to Issue 129 of the Secondary and FE Magazine

Does this sound like you at the end of last term? It certainly rang true with some of us!



We all know we should have done the opposite and that we should have opened our mental car boots and laid everything out on a tarpaulin for review and reflection, and been all Janus-like looking both forwards and backwards as we stepped through the door from 2015 to 2016, but we can be forgiven for keeping the boot firmly closed during the Christmas frenzy, the New Year Nurofen quest, and the Sunday 3 January horror of "How can it be the start of term already?"

So, instead, we @NCETM did some looking both ways on your behalf, with a particular eye for GCSE-related material: in Heads Up you can see what we found. We hope it's of use to you.

Contents

Heads Up

If a year ago you made the resolution to read more maths education blogs so as to keep up to speed with current thinking only to find that your resolve soon wavered once term was underway, @mathsjem's compilation of her ten favourite blog posts from 2015 is a great way to get the same resolution off to a much better start in 2016. There's been a number of posts in the last few months which are an invaluable support for the teachers (and their pupils) who are now four months into the new GCSE: particular thanks must go to @Just_Maths for the Herculean task of collating by topic almost all the specimen exam questions from the four exam boards. Moving beyond maths blogs, the always interesting writer Evidence into Practice has been very productive over the last few weeks, and added new posts about "Is teaching a natural ability?", "The artificiality of teaching" and "The psychology of behaviour management".

Back to maths: to help your planning for the coming term and beyond, we've been leafing through recent back issues of the NCETM secondary magazine to find the articles that link to the new GCSE:

Topic	**Link to issue & articles**
Fluency with "times tables"	128, 122 (and 121 for a global perspective)
Problem solving	128
Geometrical reasoning	114, 119, 120, 121, 125, 126, 127
Algebraic reasoning	116 (and here also), 117, 123
Multiplicative reasoning (and bar modelling)	120, 121, 127
Statistical reasoning	117, 123
Fractions	114, 115 (and here and here also), 116, 118, 119
Surds	116, 122
Order of operations	118
HCF and LCM	127, 128
Pythagoras and Trigonometry	118, 123, 125
Bearings	127
Transformations	125, 126
Linear sequences	114
Expanding binomials and factorising quadratic expressions	119
Probability and modelling uncertainty	126, 118, 117

We hope this is helpful. What have we not yet written about that you would like us to? Let us know, as always, by email to info@ncetm.org.uk or by Twitter @NCETMsecondary.

And in other news…

The full set of videos from the secondary teachers who took part in the Shanghai exchange can be watched here.

If any of your current Year 13s are thinking about becoming a school maths teacher after university, then please point them to these Future Teaching Scholars webpages. This is a new scheme, where undergraduates who commit to teaching maths (or physics) can get a £15K grant to help them through university, and specialist teacher training once they graduate. But tarry not! The deadline for this year's A level students is **31 March**.

More or Less is back on BBC Radio 4 – live on Fridays, repeated on Sundays, and downloadable thereafter. Slice of cake, mug of tea, and Tim Harford: just perfect.

Image credit
Page header by ijclark (adapted), some rights reserved

Secondary Magazine 129

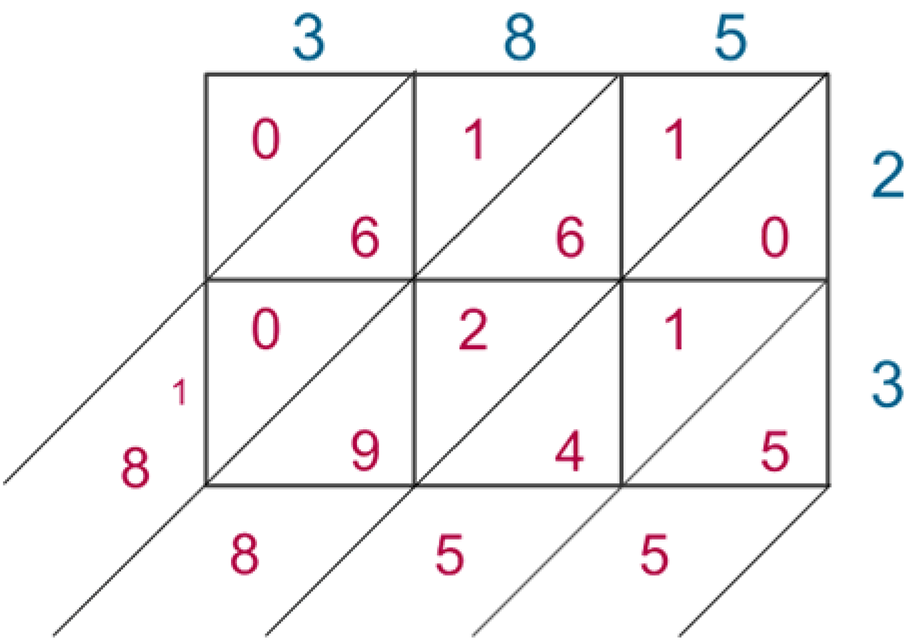
National Centre for Excellence in the Teaching of Mathematics

Building Bridges

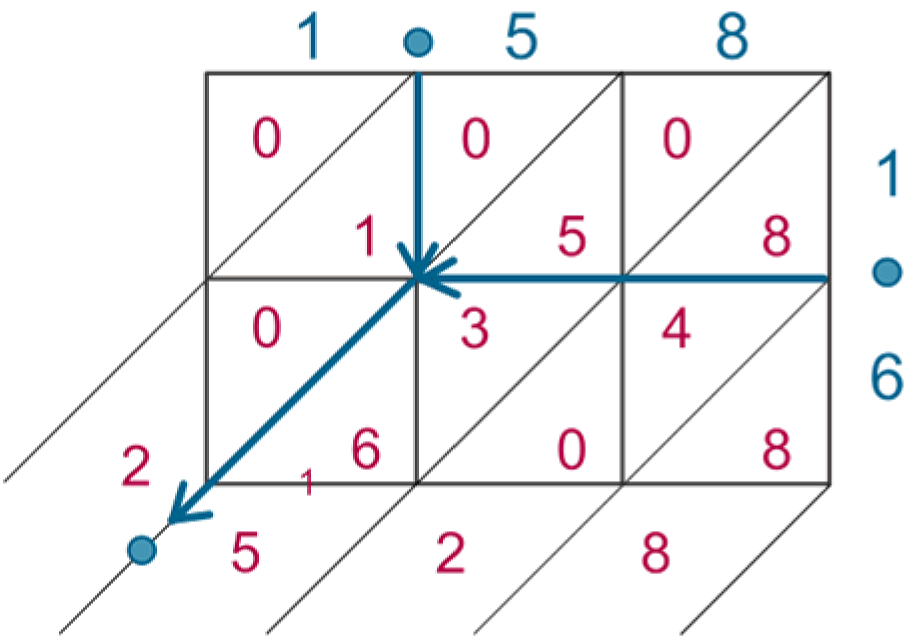
In the last issue, we discussed strategies for developing pupils' fluency with their times tables. Once these key facts are secure, how can we improve our pupils' fluency with procedures that use them, in particular "long multiplication"? We discussed last year (in Issue 119), the grid method, and advocated it as a powerful and long-lasting model that helps pupils extend their conceptual understanding from area and the multiplication of numbers to the multiplication and factorization of algebraic expressions. A challenge to the grid method is that it's not very efficient for multiplying two 3-digit or larger numbers, and that it requires pupils to be confident with place value: it's a good stepping stone from concrete to abstract multiplication, but it's often not the best method when actually you want to calculate the product of two numbers. The traditional "long multiplication" algorithm is more efficient than the grid method, but it too expects confident understanding of place value: in a KS3 class with pupils with low prior attainment, that may be an unrealistic expectation, at least when they first join the school.

An algorithm which takes away the stress of being forced to consider place value (which will result in errors if zeros are forgotten or misunderstood and hence misplaced), is to use what's often called the Lattice Method. We must point out here, for historical accuracy, that this is sometimes incorrectly referred to as Napier's Method. Napier did devise a clever system of multiplication using blocks (known as Napier's Bones) built on the Lattice Method; this is a story well worth investigating further. A strong advantage of the Lattice Method is that it requires, and hence reinforces, understanding of and fluency with times tables up to 9, and most of the place value thinking is 'below the surface'. Pupils, especially those with low prior attainment, therefore often find it an easier algorithm to implement correctly than the grid method or classic "long multiplication"; this builds their confidence, which is hugely important. The lattice method helps pupils get the right answer without getting bogged down by other concepts which may not be securely understood, or which may be understood but in practice can still produce errors.

The beauty of the algorithm is in its simplicity: let's take an example and calculate $23 \times 385 = 8855$. We multiply the digits pairwise, and record the answer to each separate multiplication using tens and units above and below the diagonal lines:



It can also be used effectively to multiply two decimals such as 1.58 × 1.6:



Of course, the decimal point can be inserted after calculating first the integer product 158 × 16, and then considering the scale factors between the original factors, the integers, the integer product, and hence the original product. There are short videos on YouTube showing lattice multiplication and decimal lattice multiplication.

The caveat to the lattice model is no doubt obvious, but let us state it nonetheless: the lattice method is a good **algorithm**, but it's **not** a good **model** for multiplication – in particular, it doesn't generalise naturally from the concrete to the abstract. Therefore, once pupils are confidently calculating products using a lattice, they must then also calculate them using a grid and a "long multiplication", so that they develop understanding of and fluency with these more powerful models.

Also, this is a good context for developing pupils' reasoning: they can consider a rich question such as "what's the same and what's different?" about the three methods as they are written out on the page – comparing the lattice above with

300	80	5	×
6000	1600	100	20
900	240	15	3
6900	1840	115	**8855**

and

	3	8	5	×
		2	3	
1	1	$^{2}5$	$^{1}5$	
7	$^{1}7$	$^{1}0$	0	
8	8	5	5	

In particular, they should look at the “partway” calculations – 6900, 1840 and 115 in the grid method and 1155 and 7700 in the long multiplication. Do these also occur, perhaps hidden, in the lattice? If not, is just a miracle that the answers are all the same (we hope!)?

You can find previous *Building Bridges* features here.

Image credit
Page header by Mark Spearman (adapted), some rights reserved

Sixth Sense

Students of mathematics toy with the concept of functions from quite an early age – how many of us were challenged at some stage to "think of a number, double it, add seven, double that, add 2, quarter that, subtract the number you first thought of, and hey presto …" – though the formal notation and language of functions often catches out even some of the best at A2. Can your top-module-scoring student explain, coherently and concisely and correctly, the difference between the co-domain and the range of a function?

Using inverse and composite functions, including formal notation, certainly crops up in the AQA Level 2 Further Mathematics Qualification, which sits alongside GCSE mathematics, and other boards have historically included some of this content at GCSE Higher tier and in Additional Maths. Under the new structure, all higher tier GCSE students will be expected to interpret reverse processes and combined processes using the formal notation of inverses and composition.

Students' lack of conceptual understanding of functional notation is unmissably apparent when they are considering transformations of graphs: very few students, even those with A* grades embarking on Further Mathematics in Year 12, have an explanation that relates $y = f(x)$ and $y = f(x+2)$ other than "it's +2 in the brackets so it's 2 to the left. Or the right. I can never remember." Even if the student on the spot can remember whether it's up, down, left, right, in, out or shake it all about, the Sixth Sense we're nurturing in AS and A2 mathematicians is the ability to justify rather than the ability to parrot-recall!

The hurdle, it seems, is in moving from informal, wordy instruction sets to formal functional notation and understanding. To tackle this, you might need to go right back to the Year 2 carpet … Even very young learners are quite able to cope with

> *A number machine takes an input, doubles it and then subtracts 3 to get an output. What's the output if the input is 4?*

since all they need is an understanding of the words and some core arithmetic skills.

Even before being introduced to algebra, many of them make a reasonable attempt at the harder question

> *If the output is 9, what was the input?*

by reversing the process step-by-step.

Composite questions are similarly straightforward when discussed in words:

> *Machine A triples an input and adds 5. Machine B squares inputs. What happens if we connect the machines together, so that the output from A goes straight into B, and input 2 into this combined machine? What happens if we connect them the other way round?*

Judicious use of "input-output diagrams" where processes are chained together is clearly helpful here too. So, what goes wrong when we formalise? These natural processes which one might describe as "easy-ish" suddenly become "ferociously hard" even for the highest attaining students. Indeed, by the time they meet

$$f(x) = 3x + 5$$

they will have spent some time studying algebra, and so their first instinct might be that this is some kind of equation and ask "can we subtract 5 from both sides?" This confusion is added to by questions such as

Given $f(x) = 3x + 5$, *solve* $f(x) = 14$

Should we use a different notation for functions? It could well be less confusing to explain and interpret

$$f : x \mapsto 3x + 5$$

at GCSE and, in the sixth form,

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ &x \mapsto 3x + 5 \end{aligned}$$

(Ah-ha, there's the co-domain!)

At GCSE and A-level, rightly or wrongly, students need to learn to read $f(x) = 3x + 5$ as a definition and $f(x) = 14$ as an equation (unless, of course, it is a definition of a constant function – students' lack of confidence is completely forgivable!).

A natural approach at level 3 is via the language and notation of sequences. For whatever reason, something about the statement

Find the first 4 terms of the sequence that is generated using $u_n = 3n + 5$

causes less (but certainly not no!) confusion. So, once students are fluent with this notation, why not try

Find the first 4 terms of the sequence that is generated using $f(n) = n^2 - 3$

to encourage fluency with functional notation?

After some practice, they will happily step away from sequences and respond to more abstract questions, such as

If $f(x) = 3x - 1$, *what is* $f(5)$? $f(1.3)$? $f\left(\frac{7}{2}\right)$?

and then

what is $f(k)$? $f(3x)$? $f(x^2 - 1)$?

A good deal of time spent developing procedural fluency with functional notation is needed before we try anything too ambitious, but there are plenty of opportunities for conceptual variation as we go along:

A function f takes an input, squares it and subtracts 7.

(a) *Complete the following table:*

Input	Output
3	
-12	
	74
	93
p	
	q^2-7
$2x$	
	y

(b) *Define the function in the form* $f(x) = \dots$

(c) *Which row in your table explains how to "undo" the effect of f?*

(d) *Define this "undo" or "inverse" process in the form* $g(x) = \dots$

and clearly this is a great opportunity to practise mental arithmetic and algebraic skills – the definition at the start of the question could be considerably more demanding. Once your students are happy with the interpretation of "*find* $f(3)$" as requiring them to find the output when 3 is used as an input for *f*, it is not too giant a leap to suggest that "*find* $f(g(2))$" requires them to find the output when $g(2)$ is used as an input for *f*, which in turn needs them to find the output when 2 is used as the input for *g*. It is worth spending time using lots of brackets before explaining that mathematicians often ditch the outer pair so that $fg(2)$ is commonly accepted to mean exactly the same thing as $f(g(2))$.

Working with composite functions is an excellent example of a scenario where specialising before generalising helps enormously: the leap from finding $fg(2)$ to finding $fg(x)$ is a large one and should not be attempted prematurely. Ultimately, though, we are aiming to get our students to see *x* here as a shorthand for "input" rather than as anything special, or unknown in the "solving equations" sense, so that when they see

Given $f(x) = 2x^2$ *and* $g(x) = 4x - 1$ *find* $fg(x)$ and $gf(x)$

they can suggest that $fg(x)$ takes an input, multiplies it by 4, subtracts 1 and then takes this whole output, squares it and finally doubles the result. That is,

$$fg(x) = f(4x-1) = 2(4x-1)^2$$

and similarly that

$$gf(x) = g(2x^2) = 4(2x^2) - 1 = 8x^2 - 1$$

All of this discussion of inputs and outputs, with *x* representing the input, may at some point remind learners of how and why we draw graphs in mathematics: often we have a relationship, typically between an input *x* and an output *y*, and we plot and join up points that meet the conditions specified by the relationship – the points whose coordinates satisfy the relationship.

Consequently the instruction

draw a graph of $y = f(x)$ *where* $f(x) = 7x - 1$

merely says "for any input value, the output value is calculated by multiplying by 7 and subtracting 1" and hence we find ourselves plotting points that satisfy the relationship y-coordinate $= 7 \times x$-coordinate $- 1$.

The language of inputs and outputs is very useful when looking at transformations of graphs: many A* students end up learning the "rules" here because they find the justification hard. But the justification is well worth it, and follows naturally from the sequence of ideas we've set out above.

Again, specialising first helps: suppose we have a graph of $y = f(x)$ in front of us, and we can see that the point (2, 7) is on the graph. This means that when we input 2 into *f* the output is 7.

What is the output if we input 2 into $3f(x)$*?*

Hopefully we get the answer "21" in unison. This means that if we drew the graph of $y = 3f(x)$ we'd have a point at (2, 21) – which would be the image of the original point at (2, 7) after some transformation.

What is the output if we input 2 into $f(3x)$*?*

This is a harder question, but hopefully we've been going carefully enough to realise that this doesn't really make sense – is 2 the whole input and thus the same as 3*x*, or is this a badly worded question that is actually asking us to find $f(6)$? Without a definition of *f*, we can't do this!

In fact the only thing we can say with any confidence is

In order to get an output of 7 from $f(3x)$*, we would need to input* $x = \frac{2}{3}$ *so that we'd be calculating* $f\left(3 \times \frac{2}{3}\right) = f(2)$ *which we know is equal to 7.*

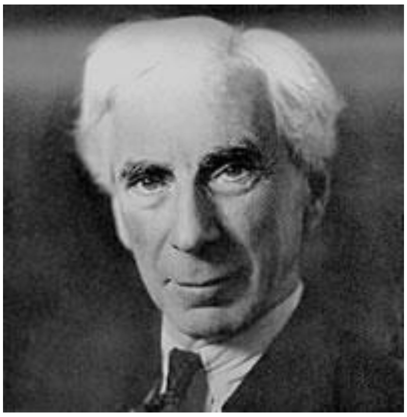
This means that if we drew the graph of $y = f(3x)$ we'd have a point at (⅔, 7) – which would be the image of the original point at (2, 7) after some transformation. After more of this "pointwise thinking", we're now in a position to conclude that

1. *If the point* (a, b) *lies on the curve representing* $y = f(x)$*, then we can be sure that the curve given by* $y = 3f(x)$ *will pass through the point* $(a, 3b)$*. That is,* ***the outputs triple****, and the geometric effect is that the graph of* $y = f(x)$ *is stretched by a scale factor of three parallel to the y-axis to become the graph of* $y = 3f(x)$
2. *If the point* (a, b) *lies on the curve representing* $y = f(x)$*, then we can be sure that the curve given by* $y = f(3x)$ *will pass through the point* $\left(\frac{a}{3}, b\right)$*. That is,* ***the inputs divide by 3****, and the geometric effect is that the graph of* $y = f(x)$ *is stretched by a scale factor of one third parallel to the x axis to become the graph of* $y = f(3x)$

Similar thinking explains the transformations between $y = f(x)$ and $y = f(x) - 5$ and $y = f(x-5)$: it takes time to embed, but it is of so much more value here than the alternative “learn these rules” approach.

You can find previous *Sixth Sense* features here.

From the Library



What is your image of a 'professional' mathematician? What do our students think a 'real-life' mathematician is like? How does this image affect our students' academic choices, is it different for male and female students, and what can teachers and schools do to counter negative images and their effects? Participation in Mathematics and Further Mathematics at A Level in England and Wales has steadily increased in recent years – good news – but the ratio of girls to boys continues to be significantly unbalanced, at about 1:2 for Mathematics and 1:3 for Further Mathematics (Smith and Golding 2015b). A number of research studies have been undertaken on participation in general, with some focussing on gender in particular, including a current four-year project run by the Further Mathematics Support Programme.

It has been widely established that there are five key factors that influence pupils' continuing with Mathematics at AS and A2 level: "prior attainment in mathematics, enjoyment, perceived competence, interest in mathematics, and awareness of the utility of mathematics for supporting access to other areas" (Smith 2014). Gender, along with factors such as ethnicity and socioeconomic status, can be considered as a significant background factor interacting with the five key ones. Mendick (whose article A Mathematician Goes to the Movies we looked at this time last year) et al (2008), via the perspective of gender, examined representations of mathematics and mathematicians in popular culture and the effects of these representations. They found the typical view of mathematicians to be old, white, middle-class, heterosexual men, characterised as having mental health problems and being socially incompetent, and "whose obsession for mathematics has colonised their entire personality" (p.18). Women's contributions "disappeared"; women were seen as appendages to 'greater' male mathematicians. Mathematics was found to be associated with masculine attributes, heroic and powerful, while outstanding abilities were 'natural' not acquired. The research found these images were held even as people recognised they were stereotypical (pp. 32 – 33). In contrast there was some evidence from 'texts' such as television programmes, advertisements, newspaper and magazine articles, websites and books of "a growing trend of young, attractive, 'smart girls'" as well as the portrayal "of mathematics as beautiful, often linked to pattern and nature" and associating the process of doing mathematics with "artistic, musical and other forms of creativity", rather than the subject being set in stone.

So what encourages girls' participation at A Level Mathematics? Smith and Golding (2015b) at the Institute of Education, UCL, as part of a continuing FMSP project on "Gender and Participation in Mathematics and Further Mathematics", reported on their initial findings through their research on 'good practice' in four schools and a Further Education college. While there were no initiatives aimed specifically at girls, three themes emerged: "pathway career thinking, robust emotional encouragement, and flexible cognitive support for working with challenge". Firstly, mathematics was promoted as a subject of "wide and multiple applicability rather than access to specific or elite courses". Secondly, teachers' confidence in their students was overtly acknowledged and relationships were established "in which all the students' feelings and ways of working were known to the teacher, and vice versa."

The researchers identified the combination of the third theme with the first two as key to success: offering "multiple and flexible opportunities to meet mathematical difficulties" and emphasising that "students should not expect single contacts in lessons to suffice to develop deep understanding." Girls valued teachers who managed lessons, or provided opportunities outside of class, to allow for "low-key conversations" to check understanding and "who were good at explaining ideas in a variety of ways, rather than just repeating the same explanation, showing the value they placed on teachers who could combine their knowledge of students with good pedagogic knowledge of mathematics". Girls enjoyed the "experiences of struggle, support and success", "but not about feeling pressured to go faster than they could understand"; "they enjoyed the experience of personal achievement coming out of strong supportive class- and teacher-relationships".

So three factors emerge as key to girls' take-up of mathematics: sound knowledge preparation, individual teacher encouragement and supportive peer relationships.

Resources

Gendered stereotyping, the research suggests, is very nuanced. The Campaign for Science and Engineering considers the message "STEM is for girls too" reinforces the STEM and gender stereotypes (CaSE 2014). Rather, a diverse range of images should be used: "young and old mathematics users, attractive and not attractive, sporty and not sporty, with a particular focus on users of average ability and career success." (Smith 2014).

The University of Nottingham has produced a number of videos where women explain how and why they took up jobs involving mathematics. A wide range of biographies of professional mathematicians is available at the Facebook page Women In Mathematics. The Institute of Mathematics and its Applications (IMA) manages the website Maths Careers which features a wide range of people using mathematics in their work.

FMSP has provided a number of resources and further details of their project Encouraging Girls to Take Mathematics, while the Institute of Education at UCL has the research project Supporting Advanced Mathematics.

In addition to the findings from the 'good practice' cases above, Mendick *et al* (2008), Smith (2014), and the FMSP Increasing Girls' Participation in Mathematics Briefing Document make a number of easy-to-implement recommendations.

References

CaSE. (2014). Improving Diversity in STEM: A report by the Campaign for Science and Engineering (CaSE). Campaign for Science and Engineering.

Mendick, H. (2007) 'I could always just play': gender and mathematical ability.

Mendick, H., Moreau, M-P. and Hollingsworth, S. (2008) Mathematical Images and Gender Identities: A report on the gendering of representations of mathematics and mathematicians in popular culture and their influences on learners. Project Report. UK Resource Centre for Women in Science Engineering and Technology, Bradford. Goldsmiths Research Online.

Smith, C. (2014) Gender and participation in mathematics and further mathematics A-levels: a literature review for the Further Mathematics Support Programme.

Smith, C. and Golding, J. (2015a) Gender and Participation in Mathematics and Further Mathematics: Interim Report for the Further Mathematics Support Programme.

Smith, C. and Golding, J. (2015b) Raising girls' participation in A-level mathematics: initial findings from 'good practice' case studies. Edams, G. (ed.) Proceedings of the British Society for Research into Learning Mathematics 35(1), 72 – 77.

You can find previous *From the Library* features here.

Image credits
Bertrand Russell in 1938, courtesy of Wikimedia Commons, in the public domain
Bindi Brook, from the University of Nottingham's Women in Maths YouTube playlist

It Stands to Reason

Solving a mathematical problem sometimes involves using ***inverse*** processes intelligently, fluently and flexibly. In Teaching Mathematics At Secondary Level, Tony Gardiner argues (page 13) that in many textbooks and exams such processes "have been neglected, or have been distorted by providing ready-made intermediate stepping-stones that reduce every ***inverse*** problem to a sequence of ***direct*** steps". He gives examples of situations in which pupils fail to recognise that they need to use the ***inverse*** of a ***direct*** procedure that they know, or what the ***inverse*** of that procedure is. For example, he writes (page 61) "A pupil may know how to 'find 75% of £120' yet fail to relate this *direct* operation to *inverse* variations, such as 'A price of £90 is raised to £120. What percentage increase is this?'."

Here we look at some aspects of, and classroom approaches to, developing pupils' reasoning about ***undoing*** mathematical transformations, in the widest sense.

Doing and undoing a one-step action
Ask pupils to give examples of one-step actions or 'doings', and to describe what action would be the 'undoing' of each one. Examples might include …

doing	***undoing***
open a door	*close the door*
put a cake in a tin	*take the cake out of the tin*
tie a knot	*un-tie the knot*

Now ask them to think of mathematical examples …

mathematical doing	***mathematical undoing***
add 3	*subtract 3*
rotate 90˚ clockwise	*rotate 90˚ anticlockwise*
factorise	*multiply-out*

Ask for another example, and another, and another … and establish the conventional vocabulary:

'direct' (action or process) means 'the doing'
'inverse' (of the action or process) means 'the undoing'

Now ask them to think of one-step actions where ***the 'undoing' is the same as the 'doing'***? With some prompting (rotating a book through 180˚ usually generates a collective "oh yes …") they should come up with, or at least agree with, examples such as …

turn upside-down
rotate through 180˚
divide ***into*** *12 (but not "divide by 12" … the distinction is worth making clear)*
subtract ***from*** *10 (but not "subtract 10")*
reciprocate (a fraction)
negate

…. i.e. **'self-inverse'** actions.

Now can they think of 'doings' (or actions) that share another special property – 'doings' that do not change whatever they act on? Pupils' mathematical examples should include …

add 0
multiply by 1
rotate through 360˚

… i.e. **‘identity’** actions.

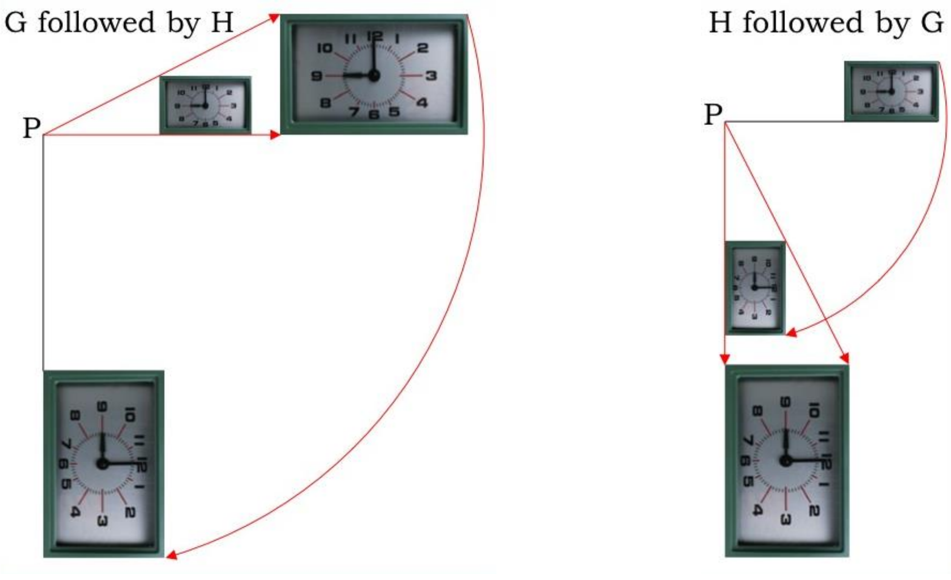
Doing and undoing a two-step action

With two-step actions the question for pupils is:

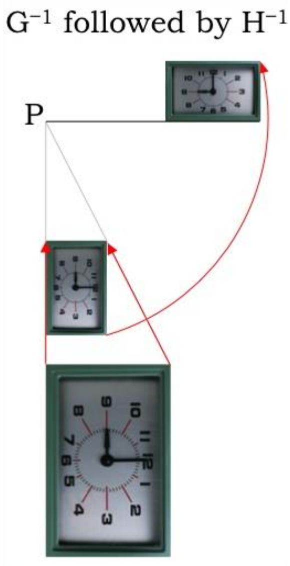
*How do you **undo** the overall, combined, effect of doing one thing after another?*

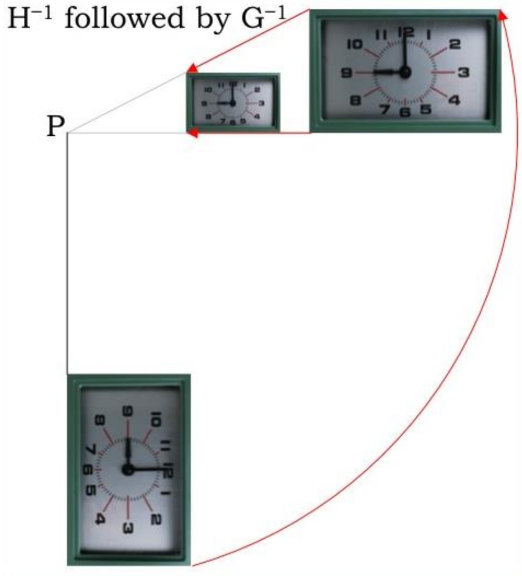
Pupils will realise that each separate step has to be ***undone***. That is, they will probably be able to say that they need to do the ***inverse*** of each step. The challenge is to get them to see for themselves that the ***inverse*** of each step must be done in the order that is the ***reverse*** of the order of the ***direct*** steps – making sure they can distinguish between the meanings of the words ‘reverse’ and ‘inverse’.

Pupils often think that the inverse of ‘A-followed-by-B’ is ‘the-inverse-of-A’ (which, for speed, we’ll now write as A^{-1}) followed by ‘the-inverse-of-B’ (or, more succinctly, B^{-1}). They are likely to make this incorrect generalisation if they start by investigating two-step actions in which the ***direct*** action ‘A-followed-by-B’ has the ***same effec***t as the ***direct*** action ‘B-followed-by-A’; this is because when, and only when, the combination of ***direct*** steps (of a two-step action) is commutative, then so is the combination of the ***inverses*** of those steps. For example, since the two separate steps, *G : enlarge ×2 with centre P* and *H : rotate 90˚ clockwise about P*, of a two-step transformation have the same combined effect whichever single step is done first …



… the inverses of those two particular single steps are also commutative under combination …





Consequently, pupils may conclude incorrectly that (G-followed-by-H) $^{-1}$ is G^{-1}-followed-by-H^{-1} because it has the same ***final*** effect as the actual inverse, which is H^{-1}-followed-by-G^{-1}. However, when all four diagrams are looked at together they show that …

the ***inverse of the whole*** action, that is (G-followed-by-H) $^{-1}$
is not the sequence of actions G^{-1}-followed-by-H^{-1}
but is the sequence of actions H^{-1}-followed-by-G^{-1}

and that …

the ***inverse of the whole*** action, that is (H-followed-by-G) $^{-1}$
is not the sequence of actions H^{-1}-followed-by-G^{-1}
but is the sequence of actions G^{-1}-followed-by-H^{-1}

Therefore, if pupils are going to generalise correctly for themselves, they need to explore right from the start two-step actions in which the two steps are **not commutative** when combined. The wider the variety of sequences of visual images representing 'real life', geometrical and arithmetical processes they see, the more secure their grasp, and recall, is likely to be.

One strategy is to show them a sequence of images depicting the ***doing*** of a two-step action (ensuring that the two steps are not commutative under combination), and then offer two alternative sequences of images representing the ***undoing*** of that two-step action. Pupils have to decide which of the two sequences truly represents the ***undoing***, and explain why. Here are three examples.

Example 1

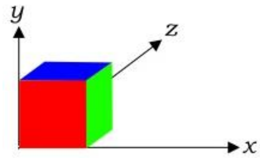

Doing
take off boots
take off socks


Undoing
put on boots
put on socks

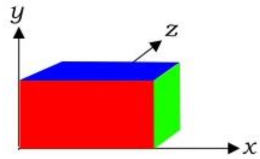

Undoing
put on socks
put on boots

Example 2

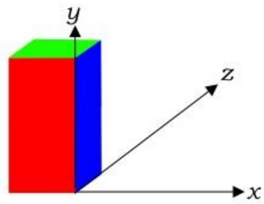
Doing



opposite faces of the cube are the same colour

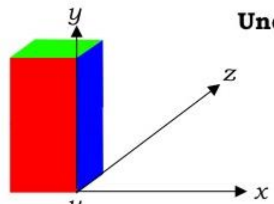


double all x-coordinates

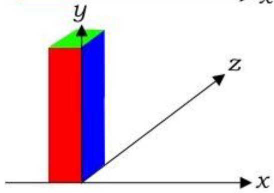


rotate 90° anticlockwise about z-axis

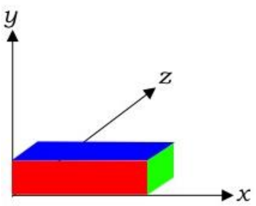
Undoing



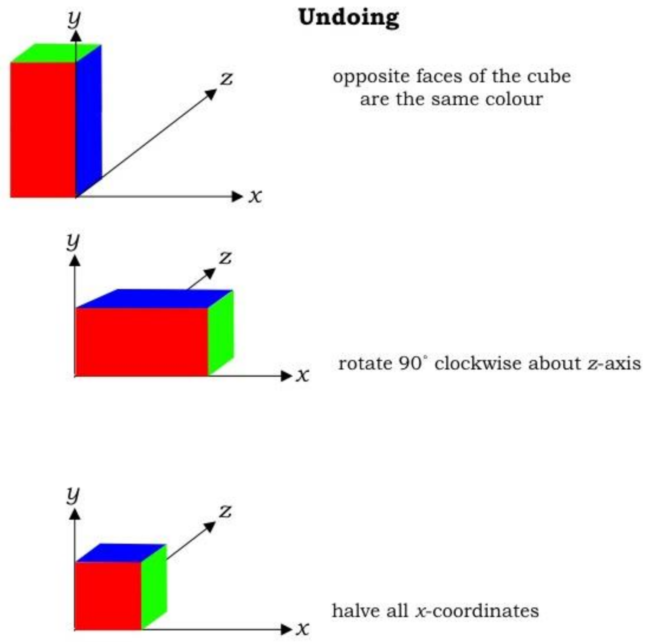
opposite faces of the cube are the same colour



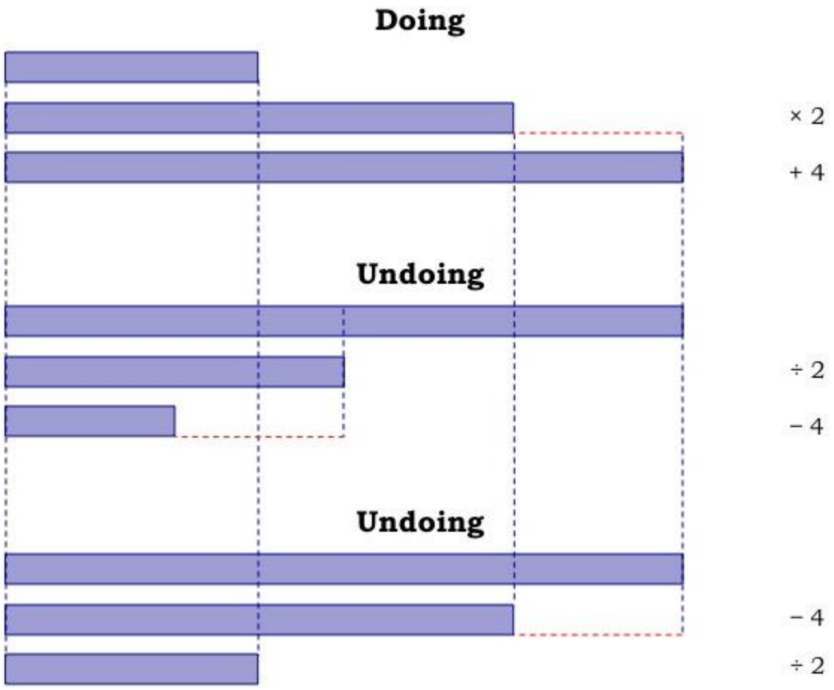
halve all x-coordinates



rotate 90° clockwise about z-axis

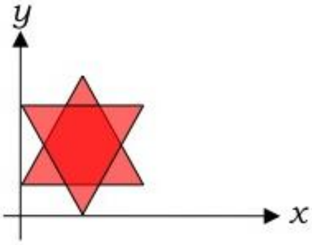


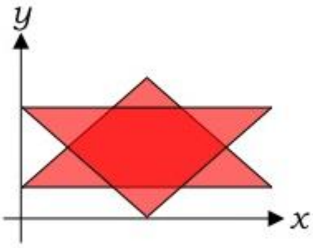
Example 3



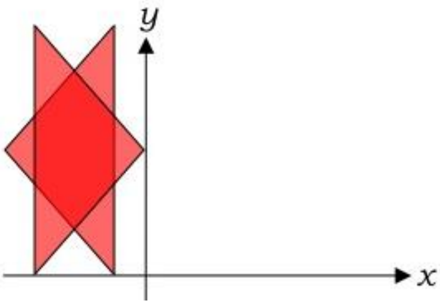
Next you could give the sequence of images showing the ***doing***, and challenge pupils to create for themselves a sequence of images showing the ***undoing***. For example, what sequence of images would show the ***undoing*** of this two-step process?

Doing





double all x-coordinates



rotate 90° anticlockwise about origin

You can find previous *It Stands to Reason* features here

Eyes Down…Misconception or Mistake?

by Mel Muldowney, a secondary school maths teacher working in the Midlands

As teachers, we come across students' work all the time that is just plain wrong. However it is important that we understand the difference between mistakes, which are generally due to carelessness (the student will have demonstrated the skill elsewhere correctly) and misconceptions, whereby a misleading idea is applied or a belief that a certain rule that applies in one set of circumstances will always apply.

Misconceptions are often deeply entrenched ideas which can prove stubborn to shift because at a cognitive level the disparity of what a student is being told conflicts with some earlier learned idea. Research suggests that teaching is more effective when misconceptions are identified, so I've been on the hunt for common misconceptions and lo-and-behold I found this example from a year 9 student:

Emma carried out a survey of the number of homeworks completed by 32 students last week.

Number of homeworks	Frequency	
0 ×	3	3 ←
1 ×	1	2 ←
2 ×	12	24 ✓
3 ×	8	24 ✓
4 ×	6	24 ✓
5 ×	2	10 ✓
	32	87

Calculate the mean.

$87 \div 32 = 2.71875$

Having spoken to the student it was evident from the response "oh yeah, obviously anything multiplied by zero is zero … what an idiot!" that one of these was a mistake brought about through lack of concentration and could have been avoided by them checking each stage of their working. When we discussed the second idea of "1 × 1 = 2" the student was adamant that it was correct and it wasn't until we did a bit of "pattern sniffing" looking at 1 × 2, 2 × 2, 1 × 3, 2 × 3 etc that a look of realisation spread across his face.

What I've learnt is that identification of misconceptions isn't enough - they need to be challenged directly with the student. Ignoring misconceptions should not be an option and to simply mark something wrong with an "×" often leaves the student knowing that something is wrong but may not necessarily leave them with the understanding required to address the matter.

If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb) to us at info@ncetm.org.uk with 'Secondary Magazine Eyes Down' in the email subject line. Include a note of where and when it was taken, and any comments on it you may have. If your picture is published, we'll send you a £20 voucher.

You can find previous *Eyes Down* features here