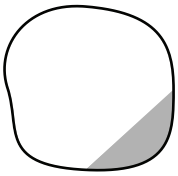
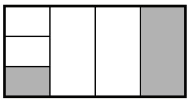
Professional development resource

Unit 1 - Deepening understanding of fractions
Lesson 1a: Parts of a shape

Lesson summary



In this lesson we think of fractions as parts of a whole, and make use of geometric representations such as shaded parts of a region (as well as using the standard numerical symbolisation). Students have the opportunity to estimate fractions in a qualitative way (eg, left) but there is also scope for increasingly analytic approaches (eg, right).



Focus of student learning

In this lesson we explore students' understanding of the part-whole view of fractions.

- How familiar are students with the part-whole view of fractions?
- In estimating a part of whole, do students take account of the size of the whole (ie of the 'unit')?
- When partitioning a shape, do students appreciate the value of having equal parts?
- How do students interpret their representations? Do they over-interpret them?
- Where students use formal procedures, can they explain why they work?

Lesson preparation

- It is important to try **Pre-task 1** or **Pre-task 2** with the class a few days before teaching Lesson 1a.
- The file **UNIT-1a-slides.pdf** contains
  - slides of the **Pre-tasks**
  - sildes of the Stage 1, 2, 3, 4 diagrams including 'cumulative' versions for Stage 1 and 2
  - a worksheet showing several copies of the Stage 3 diagram, which you might want to print out.
- The Stage 3 worksheet is also shown on page 7 (the back page) of this document.

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Lesson 1a Pre-tasks

Pre-task 1

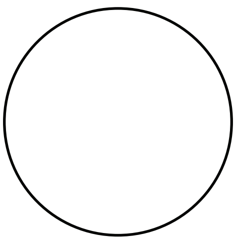
Sharing out a marshmallow tube

The picture below represents a marshmallow tube.
Show on the picture where you would cut the tube to share it between **5 people**.
Try to be as fair as possible.

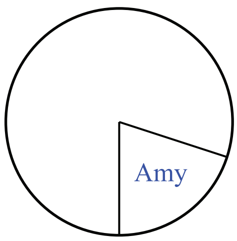
Pre-task 2

A slice of cake

This is a cake:



Amy cuts herself a slice.



Roughly how big is Amy's slice of cake? ....

Explain: ....

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Commentary: Lesson 1a Pre-tasks

The purpose of these tasks is to *harvest* students' thinking, to help with the planning of Lesson 1a. If done as a whole-class discussion, you might want to spend about 10 – 20 minutes on a task, and then start your normal lesson. It is OK to leave the task unresolved.

Pre-task 1

This can be done as homework, individual classwork or as a whole-class discussion, though it is useful to be able to observe students partitioning the tube. You might want to repeat the task for, say, 6 people.

Strategies include the following.

- Guessing the size of a piece, marking up in pieces that size and leaving the last piece too big or too small.
- Guessing the size of a piece and extending or reducing the length of the bar to make the last piece fit.
- Guessing the size of a piece; repeatedly drawing pieces of that size until the end of the bar. Rubbing out and starting again, re-adjusting the size of the piece depending on the result of the first attempt.
- Requesting to use a ruler.
- Folding. This can lead to some very accurate shares (eg for 3, 4, 8, 6, 9, 12 people.)
- Locating the middle (even when the divisor is odd) and estimating either side of the middle to identify the ends of the middle piece.
- Locating the middle (even when the divisor is odd) and putting a mark slightly to the left or right of this. Filling one side of this mark with one more piece than the other side.
- Using existing bars to derive other divisions. Eg The markers on a 3 people bar can be replicated on a 6 people bar then halve each piece.

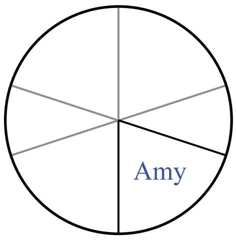
[Taken from an RME-based activity designed to help students develop meanings for fractions, by Sue Hough]

Pre-task 2

This can be done as homework, individual classwork or as a whole-class discussion.

The aim is to find out whether students

- spontaneously use fractions, decimals or percentages to describe size
- 'visualise' the amount, eg "it is a bit less than ¼, so maybe ⅕ or ⅙" [some might write ⅓ for a fraction less than ¼]
- proceed more 'analytically', eg by trying to partition the circle into (equal?) parts (be it mentally or by actually drawing lines), or by using this kind of image (below) to argue that Amy's slice is more than ⅙, so maybe ⅕.



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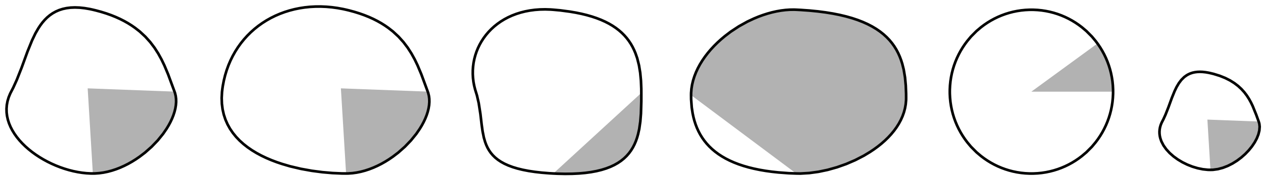
The Lesson

Stage 1: Work through this task as a class (with some pair-work if desired). Display shape A, then A and B, then A and B and C, etc. If students don't do so spontaneously, draw their attention to pairs of shapes.

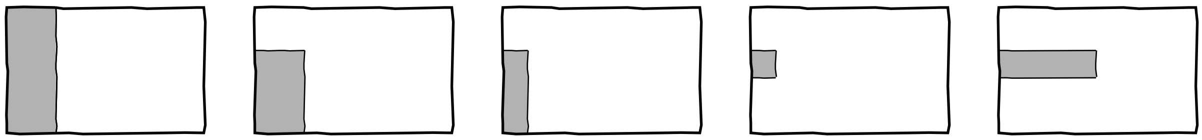
Look at these six shapes. Roughly how much of each shape is grey?



A B C D E F

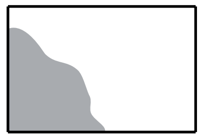
Stage 2: Work through this task in a similar way (perhaps with more group work). Display shape A, then A and B, then A, B and C, etc. If students don't do so spontaneously, focus attention on pairs of shapes.

These drawings show the same rectangle.
Roughly what fraction of the rectangle has been coloured grey in each case?



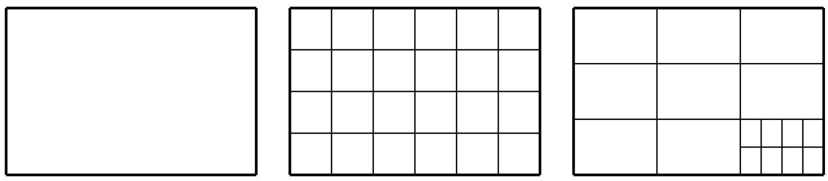
A B C D E

Stage 3: Start with X. Discuss ways of shading it, starting with 'qualitative estimates' (right), before moving to a more analytic approach (which might include partition lines). students should then work on their own or in groups, interspersed with whole class discussion.



Shade $\frac{3}{8}$ of each rectangle.

Try it in different ways.
Use the construction lines to help you.

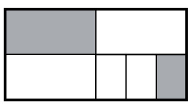


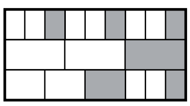
X Y Z

Stage 4, Extension: This can be done individually, or as a group or whole-class activity.

For each rectangle,
find the size of each grey region
(as a fraction of the rectangle) and
find the total amount of grey.

X


Y


Z


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Lesson commentary

It is not necessary to cover all the stages of this lesson (where practicable, omitted stages can be considered at another time). But stages should be considered in depth: be prepared to spend 10 – 30 minutes on any one stage.

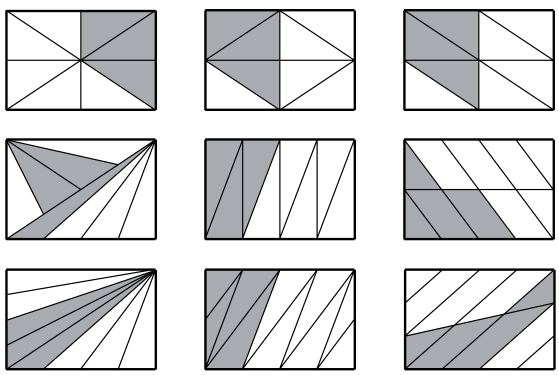
Stage 1: The diagrams are deliberately 'unstructured', to see how and to what extent students structure them themselves (which they might do by visualising, or by drawing actual lines to partition the shapes).

- Students might interpret the given phrase 'how much' in different ways. '... of each shape' suggests that we are asking for a ratio (or fraction), but some students might think we are asking for an 'absolute' amount (measured in cm^2 say). Some students might use decimals or percentages rather than fractions.
- If students don't do so spontaneously, ask them to compare diagrams:
  - eg after showing B, remind students of A (the grey region is the same but B is larger so the fraction is smaller);
  - after C remind them of A (the grey region in C is less than half that of A, so less than $\frac{1}{8}$ of C is shaded);
  - after D remind them of C (the white part of D is about the same as the grey part of C, but D is larger...).
- D is to force students away from unit fractions. E is to see how discriminating students are: can they see that it is less than $\frac{1}{8}$? (It actually shows $\frac{1}{10}$.) Shape F highlights the issue of units. Are we measuring the shaded regions against some standard, fixed, shape, or in relation to the individual shapes A, B, C, etc?

Stage 2: The diagrams should be shown 'cumulatively'. It is worth seeing whether students can tackle this mentally, ie without the distraction of trying to draw actual lines to partition the rectangle.

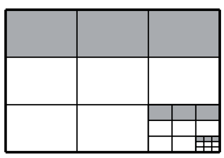
- If student invoke a rule (ie use fractional operators), can they justify this using the geometry of the situation?
- In A, about a quarter of the rectangle is shaded - students will probably readily agree with this.
- In B, about one-third of the shading has been removed. What fraction of the whole rectangle is the newly revealed small white region, and so what fraction remains shaded?
  - Some students might be able to argue along these lines: there would be 3 such white regions in the grey strip in A and we could cover the rectangle with four such strips, so we could have 3×4 such white regions in all; the grey region in B covers two such regions, so it covers $\frac{2}{12}$ of the rectangle.
  - Some students might draw actual lines and thus partition the rectangle into 12 equal parts - do they find this total by simply counting or do they derive it by structuring the situation (along the lines of '4 lots of 3')?]
- C is about one half of B; D is about one third of C; E is about 4 times as wide as D.

Stage 3: This is based on a TIMSS item. If ideas are shared and opened-up, it can lead to an infinite variety of responses (along the lines of the classic SMILE Take Half animation), thereby focussing on the notion of equal parts. You might want to produce a worksheet containing several copies of X, Y and Z.



- Z rests on the distributive law (though you might leave this implicit): $\frac{3}{8}$ of part of the rectangle, plus $\frac{3}{8}$ of the rest of the rectangle, is $\frac{3}{8}$ of the whole rectangle.
- Alternatively, $\frac{3}{9} + \frac{3}{72} = \frac{24}{72} + \frac{3}{72} = \frac{27}{72} = \frac{3}{8}$.
- Z can lead to the idea of a recursive diagram (right) and to infinite series: $\frac{3}{8} = \frac{3}{9} + \frac{3}{9\times9} + \frac{3}{9\times9\times9} + \ldots$

  or $\frac{3}{8} = 0.\dot{3}$ (in base 9).



Stage 4, Extension: This rests on the distributive law again, but students will probably be surprised that the result is $\frac{1}{3}$ each time. It can be used to challenge students who are fluent rule-users but who don't see the point of going back to childish, primary school representations ...

- You might want to encourage the use of fraction notation (eg, $\frac{1}{12} + \frac{1}{4} = \frac{1}{3}$ for rectangle A). At some stage it might be helpful for some students to copy the rectangles onto squared paper, using, say, 12×6 rectangles (but don't do this immediately).
- As a further extension, students could be asked to draw similar cases where a total of $\frac{1}{3}$ is shaded.

Adapting the lesson

- These tasks have a strong diagnostic element and should be accessible to most students.
- Depending on students’ responses, you might want to pursue tasks to a lesser or greater degree of precision, or elicit a lesser or greater variety of approaches.
- You might want to devise related tasks to focus on a particular aspect. For example, for the Stage 2 task, you might challenge students to sketch what $^{1}/_{20}$, say, of the rectangle might look like.
  “How could we check whether this was roughly right?”

Suggestions for lesson study

You might want to consider these aspects of students’ thinking:

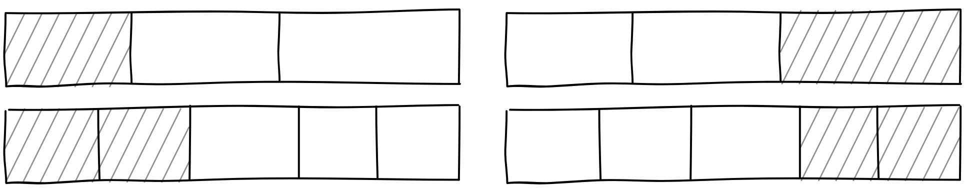
- How familiar are students with the part-whole view of fractions? Do they still make use of it?
- In estimating a part of whole, do students take account of the size of the whole (ie of the ‘unit’)?
- When partitioning a shape, do students appreciate the value of having equal parts?
- How flexible are students in finding alternative partitionings?
- How do students interpret their representations? Do they over-interpret them?
- Where students use formal procedures, can they explain why they work? In particular, can they express them using a part-whole representation?

You might also want to consider pedagogic aspects such as these:

- How did we (and how, if we teach the lesson again, could we find further ways to) elicit (and share) examples of students’ thinking?
- How did we (and how, if we teach the lesson again, could we find further ways to) build on and interrogate students’ thinking?

Research background to the lesson

- Children in the UK most commonly meet the idea of fraction (eg ⅔) as part of a whole (eg 2 pieces of a pizza that has been cut into 3 equal parts), followed by fraction as sharing (2 people share 3 pizzas). Other, related (and equally important) notions include fraction as a ratio or proportion, as a division calculation (2÷3), as an operator (⅔ of) and as a measure (eg represented by a point on a number line).
- Lessons 1a and 1b revisit the idea of part-whole. Many secondary school students will feel they have moved on from this. However, it can provide a useful support and help students make sense of rules like ‘do the same to top and bottom’ which they often find hard to explain (and easy to mis-apply).
- In these lessons we make use of representations which might be geometric or quasi ‘real life’ (eg circles or pizzas). The lessons require students to structure such representations themselves - if we do it for them, we can easily persuade ourselves that they are seeing the structure that we’re seeing when they are not. At the same time, students are liable to over-interpret their own drawings, ie to treat them as objective, leading to false arguments like “You can see that ⅖ is more than ⅓” (near right) and “You can see that ⅖ is less than ⅓” (far right).



- The main aim of the lessons is that students get a better feel for the size of a part-whole fraction by making use (mentally or by drawing) of the idea of partitioning a whole into equal-sized parts. (A complementary aim is that the teacher gets a better sense of how their students think about fractions.) The work gives students the opportunity to consider alternative partitioning’s (and hence equivalent fractions) and to relate symbolic expressions like ¼ + $^{1}/_{12}$ = ⅓, for which they might have formal transformation rules, to geometric representations.

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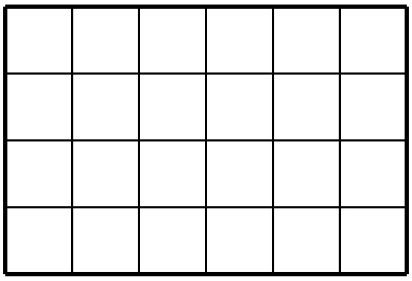
Resource: Stage 3 worksheet


X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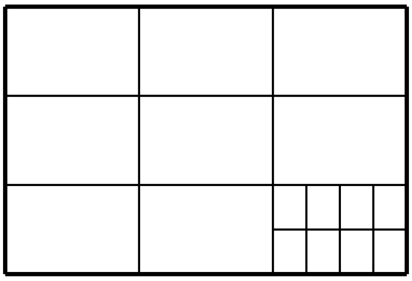
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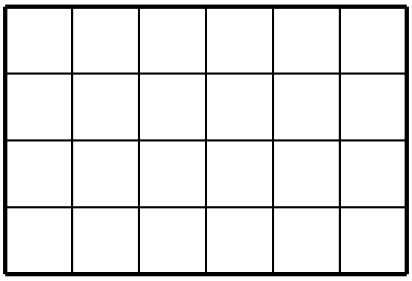
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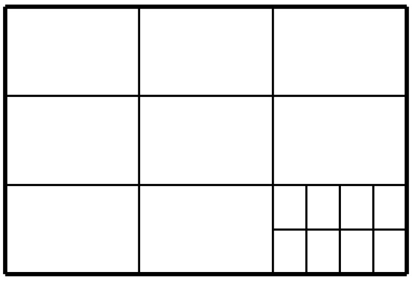
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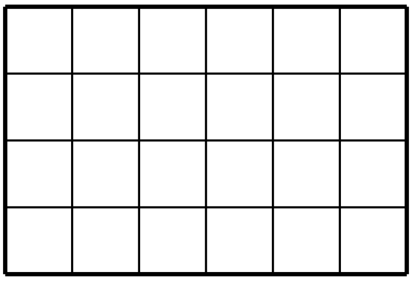
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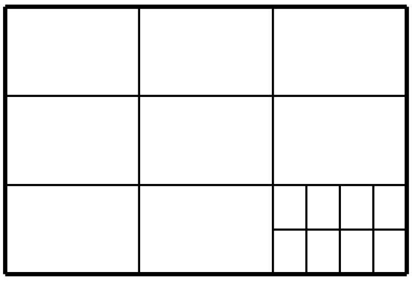
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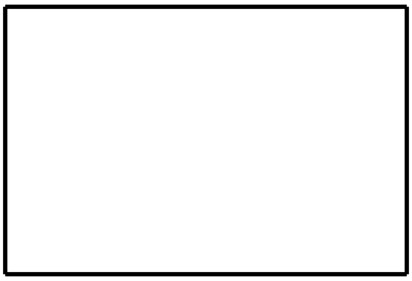
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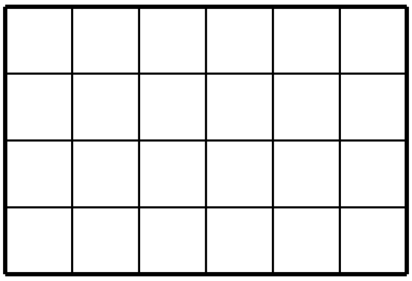
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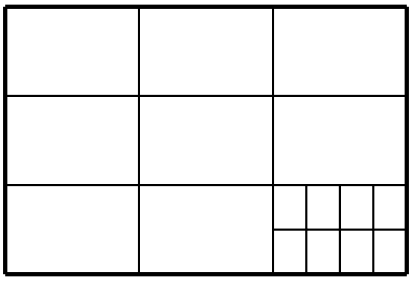
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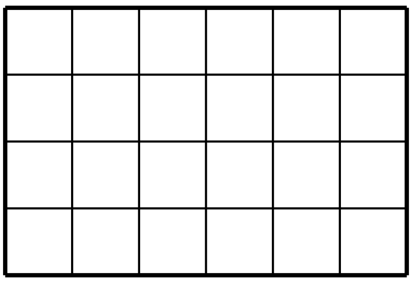
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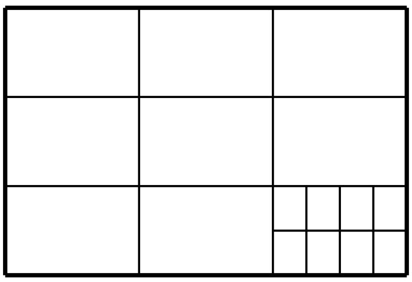
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