8 Proportional reasoning

Mastery Professional Development

8.1 Working with direct and inverse proportion

Guidance document | Key Stage 4

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme, they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'. The second of the Key Stage 4 themes (the eighth of the themes in the suite of Secondary Mastery Materials) is *Proportional reasoning*, which covers the following interconnected core concepts:

8.1 Working with direct and inverse proportion

8.2 Understanding graphical representations of proportionality

This guidance document breaks down core concept *8.1 Working with direct and inverse proportion* into two statements of **knowledge, skills and understanding**:

8.1 Working with direct and inverse proportion

- 8.1.1 Use ratio and proportion in standard measures
- 8.1.2 Use rates of change and compound measures

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

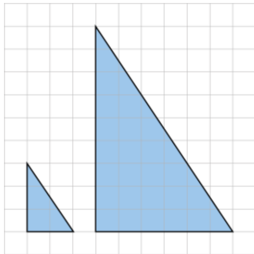
- 8.1.1 Use ratio and proportion in standard measures
  - 8.1.1.1 Understand that there are multiplicative relationships between different measures
  - 8.1.1.2 Understand how the zero property relates to a proportional relationship
  - 8.1.1.3 Identify and use the scalar multiplier between different measures
  - 8.1.1.4 Identify and use the functional multiplier between different measures
  - 8.1.1.5 Represent a proportional relationship algebraically
  - 8.1.1.6 Understand that there is a constant multiplier in all proportional relationships
  - 8.1.1.7 Use algebraic representations to calculate with complex proportional relationships
- 8.1.2 Use rates of change and compound measures
  - 8.1.2.1 Understand that fractions are an example of a multiplicative relationship
  - 8.1.2.2 Understand that compound measures are used to record a rate of change per one standard unit (using 1:*n*)
  - 8.1.2.3 Understand that compound measures are a multiplicative comparison between two quantities of different types
  - 8.1.2.4 Use the proportional relationships within compound measure to solve problems

Overview

This core concept focuses on students' maturing understanding of multiplicative relationships and extending this understanding to explore proportional relationships.

Multiplicative structures underpin much of the key content in the secondary mathematics curriculum. Direct proportion and inverse proportion refer to the set of numbers which is directly or inversely proportionally related to another set of numbers, such that each ordered pair between the sets will share the same function or constant of proportionality.

Proportional reasoning relies on students' ability to think of the numbers multiplicatively in relation to other quantities, rather than consider only the absolute value of a number. For example, thinking of 6 as 3 times as large as 2, rather than **only** as 4 more than 2.



Where a multiplicative relationship exists between two quantities, there are two possible types of multiplier: scalar and functional. Scaling suggests that the multiplier is within the same measure of quantity. For example, the base of an enlarged triangle (pictured right) is three times longer than the original, so the height will also be three time greater. Functional describes the relationship between two different measures of quantity. For example, within this family of similar triangles the height is always 1.5 times the base.

Representing proportional relationships offers insight into their structure, and a range of representations can be considered. A double number line is a powerful representation to demonstrate the difference between scalar and functional multipliers. It can also be useful for making connections to graphical representations, where one number line illustrates the domain set, represented as the x-axis; and the other number line illustrates the range set, represented as the y-axis.

Double number lines can be abstracted to a ratio table to offer a third representation for multiplicative relationships, and one that can be built on for other proportional relationships.

Students should be taught the different notation used to describe proportional relationships. The $\propto$ notation is introduced at Key Stage 4 to indicate directly proportional relationships. It is also important for students to be able to transform this representation into alternative forms. They should understand that a constant of proportionality (k) is a functional multiplier based on a multiplicative relationship between two quantities, and for all directly proportional relationships, the constant of proportionality (k) remains constant for the entire domain to range correspondence.

- Directly proportional: one quantity increases at the same rate that another quantity increases, which can be written as $A \propto B$ or $A = k\,B$.
- Inversely proportional: one quantity increases as another quantity decreases, and the relationship can be written as $A \propto \frac{1}{B}$ or $A = \frac{k}{B}$.

Examples such as $A = y \text{ and } B = x^2$ should be considered and attention drawn to the proportionality of the relationship between y and x^2 rather than y and x. Similarly, with inverse proportion, examples where $A = y \text{ and } B = x^n$ should be explored.

A key understanding is that for all directly proportional relationships, the element 0 of the domain will map to the 0 element in the range set. Because of the multiplicative nature of proportional relationships, the zero-product property applies. Contextualised examples help to demonstrate the necessity of this feature, i.e. it will cost £0 to buy 0 litres of paint. Double number lines and Cartesian graphs are both useful in drawing attention to the importance of the zero property when working with proportional relationships.

Prior learning

Multiplicative relationships underpin a huge proportion of the Key Stage 3 curriculum, so students will already have worked with multiplication in a wide range of contexts, although the language of proportion may have primarily been used when discussing ratio. Students should have a sense that any two numbers can be linked multiplicatively (except cases where 0 is the product) and recognise where a relationship between two numbers is multiplicative. They should have automatic recall of key multiplication facts, enabling them to work fluently and flexibly with numbers expressed as fractions, decimals or percentages. This recall should be supported by conceptual understanding of multiplicative structures. For example, recognising inverse relationships and knowing how different multiplication tables are related.

Students' early experiences of multiplicative relationships will have been in late Key Stage 1 or early Key Stage 2. They will have been introduced to one:many correspondence through pictorial representations or language structures such as 'for every'. Throughout Key Stage 3, students will have linked this to formal ratio notation and be able to express and interpret a situation using ratio, fractions or percentages. By the end of Key Stage 3, students may have used various representations as part of their learning around multiplicative relationships, including bar models, double number lines and ratio tables.

The range of contexts in which students will have met multiplicative relationships is vast and includes (but is not limited to) area, compound measures (for example, speed/distance/time), conversions, trigonometry and similarity. Students may not always recognise that the same mathematical structure underpins all these seemingly separate 'topics', and so work at Key Stage 4 should continue to focus on making these connections explicit. This is particularly the case as students explore different representations of proportional relationships, including graphical and algebraic representations. Students will need to make links to their prior work on manipulating algebraic equations and interpreting graphs, while identifying what is the same and what is different about these representations in the case of proportional relationships. For example, recognising that the graphs of proportional relationships always pass through the origin, and so identifying the key difference between the general algebraic representation of a straight line, $y = mx + c$, and of a directly proportional linear relationship, $y = kx$.

The core concept document *3.1 Understanding multiplicative relationships* from the Key Stage 3 PD materials explores the prior knowledge required for this core concept in more depth.

Checking prior learning

The following activities from the NCETM secondary assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Secondary assessment materials page 32	The n^{th} term of a sequence is $5n + 1$. The 0^{th} term of a sequence is 51. Albert says, 'In that case the 100^{th} term must be 510.' Is Albert right? Explain your reasoning.

<table>
<tr>
<td>Secondary assessment materials page 33</td>
<td>
All the numbers in Set 1, 2 and 3 are in direct proportion. Complete the missing entries.
<table>
<tr><th>Set 1</th><th>Set 2</th><th>Set 3</th></tr>
<tr><td>8</td><td></td><td>12</td></tr>
<tr><td></td><td>18</td><td></td></tr>
<tr><td>5</td><td>20</td><td></td></tr>
<tr><td></td><td>88</td><td>33</td></tr>
<tr><td>6</td><td></td><td></td></tr>
</table>
Could you complete the grid if any of these numbers were deleted?
Which ones and why?
</td>
</tr>
<tr>
<td>Key Stage 3 PD materials document ‘3.1 Understanding multiplicative relationships’, Key idea 3.1.1.1, Example 1</td>
<td>
Put the following products in order of size from smallest to largest.
a) $1\,142 \times 2$
b) $1\,142 \times \frac{19}{9}$
c) $\frac{7}{9} \times 1\,142$
d) $1\,142 \times \frac{9}{7}$
e) $1\,142 \times \frac{17}{9}$
f) $2\,284 \times \frac{4}{9}$
</td>
</tr>
<tr>
<td>Key Stage 3 PD materials document ‘3.1 Understanding multiplicative relationships’, Key idea 3.1.1.1, Example 4</td>
<td>
Which sets of numbers show a proportional relationship? Explain your choices.
a)
<table><tr><td>$\frac{1}{2}$</td><td>1</td></tr><tr><td>3</td><td>6</td></tr></table>
b)
<table><tr><td>1</td><td>1.1</td></tr><tr><td>2</td><td>2.2</td></tr></table>
c)
<table><tr><td>3</td><td>4</td></tr><tr><td>7</td><td>8</td></tr></table>
d)
<table><tr><td>2</td><td>4</td></tr><tr><td>6</td><td>8</td></tr></table>
e)
<table><tr><td>8</td><td>4</td></tr><tr><td>16</td><td>8</td></tr></table>
f)
<table><tr><td>4</td><td>3</td></tr><tr><td>1</td><td>0.75</td></tr></table>
</td>
</tr>
</table>

Key vocabulary

Key terms used in Key Stage 3 materials

- proportion
- ratio

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
compound measures	Measures with two or more dimensions. Examples: speed calculated as distance ÷ time; density calculated as mass ÷ volume; car efficiency measured as litres per 100 kilometres; and rate of inflation measured as percentage increase in prices over a certain time period.
direct proportion	Two variables A and B are in direct proportion if the algebraic relation between them is of the form $A = kB$, where k is a constant. The graphical representation of this relationship is a straight line through the origin, and k is the gradient of the line. Commonly, A and B are replaced by y and x, but proportionality holds when $A = y$ and $B = x^n$; y is said to be directly proportional to x-squared (or cubed etc.)
functional/scalar multiplier	In a multiplicative relationship between two quantities there are two possible types of multiplier: scalar and functional. The scalar multiplier is within the same measure of quantity. For example, in a recipe it would be the multiplier needed to scale up the quantity of a single ingredient. In a double number line representation, it would be the multiplier used **along** the number line; when considering equivalent fractions, it would be the multiplier used between two numerators or two denominators. The functional multiplier describes the relationship between two different measures of quantity. For example, in a recipe it would be the multiplier needed to describe the proportional relationship between different ingredients. In a double number line representation, it would be the multiplier used **between** the lines; when considering equivalent fractions, it would be the multiplier used between the numerator and denominator of a single fraction.
inverse proportion	Two variables A and B are inversely proportional if the algebraic relation between them is of the form $AB = k$ where k is a constant; an alternative form of the equation is $A = \frac{k}{B}$. The relations are valid for $A \neq 0$ and $B \neq 0$. If A is inversely proportional to B then A is directly proportional to $\frac{1}{B}$. In the graphical representation of this function, the axes are asymptotes to the curve. Commonly, A and B are replaced by y and x, but inverse proportionality holds when $A = y$ and $B = x^n$; y is said to be inversely proportional to x-squared (or cubed etc.)

Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

8.1.1 Use ratio and proportion in standard measures

Central to the work with proportion in standard measure is that of the unit measure. The notion of **per** relies on students having a secure understanding of unitising. Students will have met the idea of comparing quantities, such as value for money, at Key Stage 3. This should be underpinned by an understanding of the need to compare like with like. The unitary method of writing ratios in the form $1{:}n$ offers a useful demonstration of standard unitising.

The important idea of the zero-product property is perhaps most intuitive in standard measures. Understanding that £0 is worth $0, or that a soup recipe for zero people will require zero ingredients can help to contextualise this property.

Standard measures are also a useful context for highlighting the distinction between scalar and functional multipliers. Consider a currency conversion, where £1 = €1.25 and the equivalent value of £3 is required. The functional multiplier **between** measures is $\times$1.25; this is the constant of proportionality, and it is fixed for all members of the related domain and range sets. To find the equivalent of £3, the value should be multiplied by 1.25, resulting in €3.75. Alternatively, a scalar multiplier of 3 can be used to scale £1 to £3; the equivalent scaling for euros would be to also multiply €1.25 by 3, resulting in €3.75.

In this case the functional multiplier is $\times$1.25, and the scalar multiplier is 3. For another example using the same conversion rate, the functional multiplier would remain the same, but the scalar multiplier would be different.

When two variables are directly proportional to each other, the algebraic form of the relationship is $A = kB$. We can manipulate this for two variables to find k, and then use k to generate unknowns in the related set. When two variables are inversely proportional to each other, the algebraic form of the relationship is $AB = k$. We can manipulate this for two variables to find k, and then use k to generate unknowns in the related set (where $A \neq 0$ and $B \neq 0$).

There is much cross-over of terminology. Students should be clear on the distinction between k, the multiplicative constant, and c, the additive constant. Reducing the language to just 'constant' may lead to confusion.

8.1.1.1 Understand that there are multiplicative relationships between different measures

8.1.1.2 Understand how the zero property relates to a proportional relationship

8.1.1.3 Identify and use the scalar multiplier between different measures

8.1.1.4 Identify and use the functional multiplier between different measures

8.1.1.5 Represent a proportional relationship algebraically

8.1.1.6 Understand that there is a constant multiplier in all proportional relationships

8.1.1.7 Use algebraic representations to calculate with complex proportional relationships

8.1.2 Use rates of change and compound measures

Compound measures are a subset of functional relationships where different quantities are being compared, specifically two or more different types of measure of quantity. The functional relationship between kilometres and miles exists, but this is not an example of a compound measure because the same type of measure of quantity is being compared.

This core concept considers compound measures as a multiplicative comparison between two different measures of quantity. Descriptors, such as speed and pressure, are used to label this multiplicative comparison.

Much of the work on compound measures relies on a secure understanding of the functional multiplier discussed above. Students need to understand that linear functions require two pairs of values to determine the gradient or rate of change. However, for a directly proportional relationship, we can assume one pair to be (0,0) due to the zero-product property.

Students will need to appreciate the role of unitising and standard units in rates of change, including the notion of ‘per standard unit’. Compound measures are the embodiment of this per standard unit relation between measures of two different types.

8.1.2.1 Understand that fractions are an example of a multiplicative relationship

8.1.2.2 Understand that compound measures are used to record a rate of change per one standard unit (using 1:*n*)

8.1.2.3 Understand that compound measures are a multiplicative comparison between two quantities of different types

8.1.2.4 Use the proportional relationships within compound measure to solve problems

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have, and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

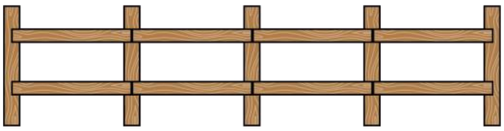
In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

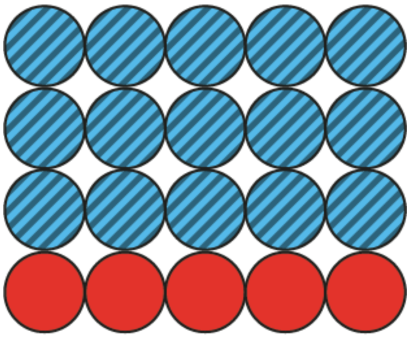
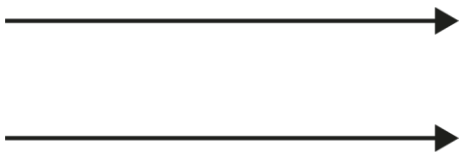
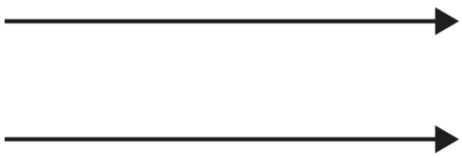
These are indicated by this symbol.

8.1.1.2 Understand how the zero property relates to a proportional relationship

Common difficulties and misconceptions
Students may overgeneralise and assume that proportional relationships refer to any linear graph, especially if their experience of proportion is limited to 'one increases as the other increases'. A distinguishing feature is that (0,0) must be present for situations involving direct proportion. A common misconception, when considering linear sequences and graphs, is to think that the value of the 50th term is 10 times the value of the 5th term. While this does apply to direct proportion, it does not apply to all linear sequences and graphs. It is important to identify this as a special case to avoid muddled thinking with more general work on sequences and graphs. If directly-proportional relationships have been experienced primarily in an algebraic form, students may have difficulty in appreciating the importance of the zero-to-zero correspondence; using a range of representations will help support the understanding of this concept. Using contextualised examples of directly proportional relationships will also make it more accessible to students. For example, understanding that when scaling a recipe, 0 g of oats will be needed to make flapjack for zero people. When determining the multiplicative constant for direct proportion, only one pair of values is required. Help students to make connections and avoid confusion by contrasting this with the two pairs of values required to calculate the rate of change with a general linear graph, and identifying that with direct proportion it could be considered that the second pair of values is (0,0). Students may find it difficult to conceptualise that, when the zero-product property is applied to inversely-proportional relationships, it creates the need to divide by zero. They may be familiar with 'dividing by zero' producing an undefined result and should now have opportunities to explore the

asymptotic nature of graphs of inversely proportional relationships, where the graph will never touch the axes.

Students need to	Guidance, discussion points and prompts
Appreciate the zero-product property *Example 1:* *Calculate:* $1 \times 0 \times 6 =$ $0 \times 0.0013 =$ $0 \times 179 =$ $-654 \times 0 =$	*Example 1* uses **variation** to draw attention to the zero-product property, using different types of number to ensure that students have experience of this property in a range of different calculations. The **language** of factor × factor = product might be familiar, and it may be useful for students to connect their understanding of that format to the zero-product property. Students might be supported to create generalisations, such as, '*If one factor is zero, the product will be zero.*' Or, '*If the product is zero, either (or both) of the factors will be zero.*'
Example 2: *Look at this multiplication.* $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 0$ *Mike says, 'I know my tables! I can work this out really quickly!'* *Explain why Mike doesn't need to use his tables and the calculation is much easier than it might first appear.*	*Example 2* gives an opportunity for building fluency with mathematical structure and **deepening** understanding of the zero-product property in a calculation that first appears complex. Developing mathematical **language** by considering the associative property may also support understanding. Discuss with colleagues to what extent you believe the language of zero-product property is necessary for students. Is this terminology that you have used in the past with your classes? Why or why not?
Appreciate the zero-product property in directly-proportional contexts *Example 3:* *Look at this fence.*  *There are 5 vertical fence posts and 8 horizontal planks.* *a) Explain why a fence made from 10 vertical posts will not need 16 horizontal planks.* *Mo uses the rule 'Start at 2 and add 3' to write a number sequence:* *2, 5, 8, 11, 14, 17, 20, 23, 26, 29…* *The 10th number in Mo's sequence is 29.* *b) Explain why the 100th number is not 290.*	The **variation** in *Example 3* gives examples (part c) and non-examples (parts a and b) of proportional situations to draw attention to the importance of the zero-product property. Students are likely to have an intuitive understanding of the property. This example is intended to bring that intuition to the fore and offer opportunities to discuss and formalise what students might already know. The **language** of 'per' or 'for every' might be particularly helpful in part c, where there are three stripey blue counters per/for every red counter and so, for zero red counters there will be zero blues. Discuss these examples with your colleagues. Do they exemplify all the contexts in which misconceptions about the zero-product property can creep in? Can you create any other examples or non-examples of the zero-product property? Could your students?

Look at the pattern of counters below.  *There are 5 red and 15 stripey blue counters.* *c) Explain why a pattern made with 50 red counters will have 150 stripey blue counters.* *d) How are the examples used different? Why does doubling the value of the 5th item in a sequence give the value for the 10th item in some situations, but not in others?*	
Example 4: *80 cm of ribbon costs £2.44.* *a) Represent this information on the double number line below.*  *b) What else do you know? For which other quantities of ribbon could you work out the cost?* *300 g of pasta is required to make dinner for 3 people.* *c) Represent this information on the double number line below.*  *d) What else do you know? For which other numbers of people could you work out the pasta required?*	The context has been varied and the **representation** has been maintained. Recognising the zero-product property in different contexts should draw students' attention to it as a necessary feature of all directly-proportional relationships. The double number line provides a format for students to make sense of the structure of the mathematics. The 0,0 aligned pair on these double number lines has been omitted. Students should recognise the importance of this point and, while they should be supported to recognise it is as necessary, it might offer a useful assessment strategy to first offer the question to a class without prompting whether zero can be placed. The **variation** in the numbers between parts a/b and c/d is intended to lead to different strategies. In the ribbon example, the values of 80 cm and £2.44 lend themselves to doubling and halving and so working along the double number line, exploring the scalar multiplier. The pasta example uses 300 g for 3 people which may lead to a scalar method (600 g for 6 people) but is perhaps more likely to draw students to a functional relationship of 100 g of pasta per person. Notice the numbers chosen in this example and throughout this key idea. Would the tasks be more effective if a different set of numbers were chosen? Experiment with different numbers: does a change in the numbers mean that your attention is drawn to similar or different things each time?

Recognise the features of directly-proportional relationships in different representations

Example 5:

8 kilometres is directly proportional to 5 miles.

a) *Write down some other pairs of equivalent distances that you know from this statement.*

b) *Represent this relationship with each of the following:*

- *double number line*
- *ratio table*
- *graph*
- $y = kx$

Ask questions such as, '*Is the zero-product property visible in this representation? Where?*' and '*Is the constant of proportionality visible is this representation? Where?*'. This will be useful in making connections between **representations** explicit to students and so **deepening** their understanding of the zero-product property. Students should be encouraged to evaluate the effectiveness of each representation in exposing the properties of directly-proportional relationships.

Consider how students might represent this relationship using axes, and the two different graphs they might produce depending on whether they define x as miles or kilometres. How might you ensure students understand that both are valid representations of the relationship? Note also $y = kx$ that is explored more deeply in key idea 8.1.1.5, so you may wish to revisit this example after further exploration of algebraic representations of proportional relationships.

8.1.1.5 Represent a proportional relationship algebraically

Common difficulties and misconceptions

The algebraic representation of a proportional relationship makes use of the functional multiplier, rather than the scalar. It is common for this multiplier to be harder for students to discern. Investing time in using representations such as ratio tables or double number lines, which make both functional and scalar multipliers explicit, is worthwhile.

It is a common misconception that all linear relationships are proportional. Comparing linear functions of the form $y = mx + c$ and $y = mx + 0$, and focusing on the key features of each in both the algebraic and graphical representations will help to address this, as will a focus on the zero-product property as in 8.1.1.2.

Equally, students may think that only linear functions can be directly proportional. Exposure to graphical representations alongside their algebraic equivalent will provide opportunities to evaluate the features of proportional relationships, with careful attention paid to which variables are being compared.

The range of algebraic symbols used when considering similar concepts can be confusing. For example, to describe straight-line graphs we conventionally use $y = mx + c$; for proportional relationships, we use $y = kx$; for linear sequences, we use $an + b$. Drawing attention to the different letters used and making explicit the connections they share, using prompts such as, '*What's the same and what's different?*' will help to give clarity.

Students might think that a multiplicative constant, k, can only be a positive integer. This stems from having only experienced proportional relationships represented graphically in the first quadrant. They should graph proportional functions with a variety of types of multiplicative constant and become familiar with the general concept of gradient as 'rate of change' between two variables.

Considering real-life examples of proportional relationships and their graphical representations will better enable students to understand both non-linear and negative proportion.

Students need to	Guidance, discussion points and prompts
Understand the algebraic form of direct proportion, $y = kx$ *Example 1:* a) *Write an algebraic expression to represent each of these relationships, defining what each symbol represents.* (i) *The total cost of buying paint which costs £3 a litre* (ii) *The value of your holiday budget when each £1 is worth €1.25* (iii) *The total distance travelled when travelling for two hours at a speed of 30mph* (iv) *The total cost of p items at £x per item* b) *Using your expressions, calculate:* (i) *The cost of four litres of paint* (ii) *The number of euros received if £2.50 is converted* (iii) *The distance travelled after eight minutes* (iv) *The cost of seven items*	Prompt students to discuss and discern the benefits of the algebraic **representation** of proportional relationships. Support them to understand that all directly proportional relationships can be represented as $y = kx$. They should recognise that each of their expressions is a form of $y = k$. Discuss with colleagues which key features of proportional reasoning are best exposed using an algebraic representation? Might the **language** of ‘per ‘or ‘for every’ support students in making fewer errors when writing equations?
Example 2: *In a bag there are six red beads and four blue beads.* *Complete the table below with simplified notation for each relationship.*	*Example 2* uses different **representations** of the same relationships to support students to make connections and appreciate how these different relationships are connected to each other. You might like to supplement these written representations with a picture or pictures. What are the benefits of working just with the symbols as the question does here? What are the benefits of adding in image(s)?

	Red to blue	***Blue to red***	***Red to bag total***	***Blue to bag total***
$\_ : \_$				
fraction				
$y = kx$				

Recognise the algebraic form of direct linear proportion.

Example 3:

Which of these equations represents a directly-proportional relationship? How do you know?

a) $y = 2x + 1$

b) $y = 2x$

c) $y = 2 + x$

d) $y = 2\sqrt{x}$

e) $y = \frac{x}{2}$

f) $y = \sqrt{2}x$

g) $y = x^2$

h) $y = x^2 + 1$

i) $y = 2^x$

j) $y = 2x^3$

Encourage students to examine the equations in relation to the general algebraic form of direct proportion, $y = kx$.

A proportional relationship has been shown here in a variety of forms and with different types of coefficient (a form of **variation**). This exposes students to both typical and atypical representations, allowing them to see what the essential features of the relationship are. Part f may cause some discussion, as students might be tempted to consider the graph of $\sqrt{x}$, rather than recognising that, here, the surd only contains 2 and so it is just a number being used as the multiplier k. Students might find it easier to recognise $\sqrt{2}$ as a coefficient if it is written as $\sqrt{2} \times x$.

Recognise where direct proportion exists in non-linear relationships

Example 4:

Calculate the area of each circle with given radius.

Use $A = \pi r^2$.

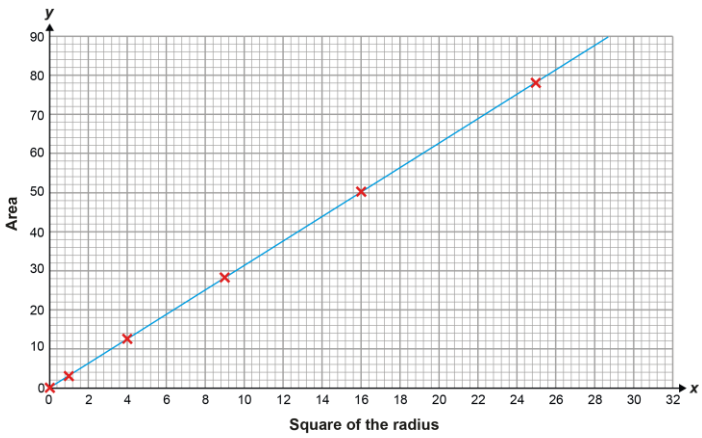
Radius	Radius squared	Area
1		
2		
3		
4		
5		

Anne says, 'The area of a circle is directly proportional to its radius.'

Do you agree with Anne?

The context of finding the area of a circle will be familiar to students from Key Stage 3. This example is an opportunity for **deepening** students' understanding, to make connections with the familiar context and link this to direct proportion. Students need to be satisfied that, while there is a relationship between radius and area, the directly-proportional relationship is between radius squared and area.

You can make a further connection to the general $y = kx$ by using a Cartesian graph as a **representation** and plotting the square of the radius against the area of the circle. Students may be surprised to see that not only is this a straight line, but also that the gradient of the line is the constant of proportionality which, in this case, is π. It is important to clarify, here, that they are using r^2 as a variable, rather than r, which is why the linear relationship exists.



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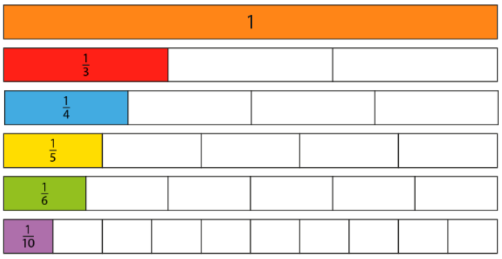
Understand the algebraic form of inverse proportion $y = \frac{k}{x}$

Example 5:

Here is part of a fraction wall.



Imagine the fraction wall continues infinitely.

a) *Describe what will happen to the size of the fractions in each row.*

b) *What will happen to the denominators of the fractions in each row?*

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Revisiting a fraction wall offers a familiar **representation** through which to develop students' understanding of inverse proportion. Students may have an instinctive awareness of one value increasing while the other decreases, but not necessarily have formalised this.

When describing the fractions on the wall, students may have used **language** structures such as, 'As the size of the denominator increases, the size of the fraction decreases.'

Draw students' attention to unit fractions as an example of inverse proportion, including the notions of relative size and 'in relation to' the one. Students will need to be secure with the idea of a procept, that $\frac{1}{7}$ is both the instruction to operate (divide one by seven) and the result of that operation.

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Example 6:

The area of a rectangle is 24 cm²^2.

a) *What could its dimensions be?*

b) *Once you have found one pair, find another, and another.*

Kathryn says, 'I noticed that some of my rectangles went together; 8×3 is like 4×6. I've doubled one length and halved the other.'

c) *From your set of rectangles with area 24 cm^2, can you find any other pairs which go together like Kathryn's?*

d) *If one length is x and the other is y, can you write a formula to show the relationship?*

e) *Plot the pairs of lengths on a graph, with one length on each axis? What do you notice?*

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Much like the fraction wall in *Example 5*, the context of finding lengths to create an area in *Example 6* gives students some familiarity to connect to the challenging idea of inverse proportion.

A **language** structure may be a helpful step to support students before they attempt to write a formula in part d. Encourage students to describe the relationship in different ways, using prompts such as:

- '*What is the product of both lengths?*'
- '*What happens to one length when you halve the other?*'
- '*What happens to one length when you treble the other?*'

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Example 7:

It takes 12 workers four hours to complete a task. Show this relationship using each of these representations:

- $\propto$ *notation*
- *algebraic notation*
- *a table of values*
- *a graph*

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The purpose of using multiple **representations** here is to draw students' attention to the commonalities between the representations and highlight the necessary features of an inversely proportional relationship.

Discuss with colleagues how the different representations each offer a different insight into the relationship here. Are there any other representations that you might choose to explore with students to help support their understanding? How about different contexts?

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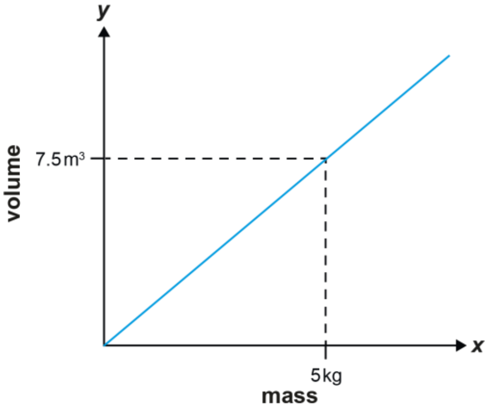
8.1.2.3 Understand that compound measures are a multiplicative comparison between two quantities of different types.

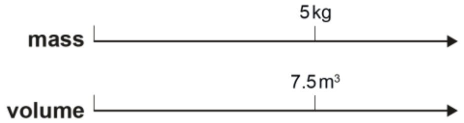
Common difficulties and misconceptions
It is a common misconception that compound measures are different from other forms of proportional relationship, but it should be understood that they follow exactly the same rules as all other multiplicative relationships. As with all multiplicative relationships, it is important for students to appreciate the difference between functional and scalar multiplicative relationships in the context of compound measures. Students need to understand the units of compound measure as a rate **per standard unit**, so spend time securing understanding of 'per' and/or 'for every' to support this. Common compound units such as 'miles per hour' are often not well understood, so break these linguistic terms down to show exactly what is meant. The use of non-standard measures, such as 'hours per mile' or 'seconds per metre' will help to deepen students' understanding and clarify their thinking. When students' experience of compound measures is primarily concerned with calculating, they are unlikely to appreciate the comparative nature of compound measures. Ensure they recognise that a compound measure refers to how one thing changes with respect to another.

Students need to	Guidance, discussion points and prompts
Understand the notion of 'per' and 'for every' *Example 1:* *Molly uses 330 g of flour to make pancakes for three people.* *a) How much flour would Molly need to make pancakes for seven people? How do you know?* *Ollie uses 340 g of flour to make pancakes for three people.* *b) How much flour would Ollie need to make pancakes for seven people? How do you know?* *Polly is also making pancakes. She uses 560 g of flour to make pancakes for seven people.* *c) How much flour would Polly need to make pancakes for three people? How do you know?*	The **variation** in *Example 1* draws students' attention to the importance of calculating a value 'per' or 'for every' unit (in this case, per person). The numbers used in Molly's pancake are easily (and probably intuitively) scaled up to seven people, so students may not be aware that they are using the fact that there is 110 g of flour per person. The shift in value for Ollie's pancake questions why that situation is more challenging and allows for reflection and formalisation of the intuitive method used earlier. Parts a and b require students to 'scale up' the amount of flour needed, whereas part c requires a reduction in value. Emphasising the **language** of 'per' or 'for every' gives students another representation to access the structure of the proportional relationship.

Example 2:

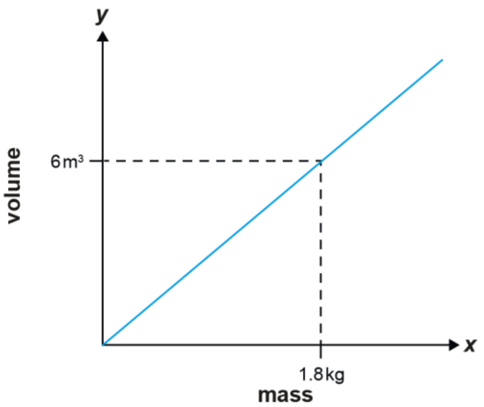
7.5 m^3 of cardboard packaging weighs 5 kg. This is shown on the graph and double number line below.

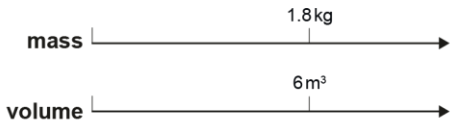




a) *Given this information, what else do you know?*

6 m^3 of airbag packaging weighs 1.8 kg. This is shown on the graph and double number line below.





b) *Using this information, what other values do you know?*

Some **representations**, more than others, demonstrate the notion of 'per' and 'for every'. Here, the Cartesian graph and double number line have been used.

Which representation do you feel is more effective in supporting students to appreciate the notion of 'per' and 'for every'? Why? Is it context-specific, or is this representation generally better for seeing this element of the structure of the relationship?

Example 2 gives an opportunity for **deepening** students' developing understanding of compound measures as one measure 'shared' between each standard unit of the other measure. In part a, students might initially find the volume per 1 kg of packaging and then multiples of per 1 kg. In part b, they may find the mass per 1 m^3 and then multiples of per 1 m^3. Encourage students to evaluate their thinking. Ask: '*How could you compare cardboard to airbag packaging*?' Draw their attention to the need to compare like with like.

Notice the numbers chosen in these examples. What is the purpose of this task? Would the task be more effective if a different set of numbers were chosen? Would students approach the task differently?

Understand the general form of compound measures $k = \frac{y}{x}$ *Example 3:* *The men's world record for running 100 m (set in 2009) is 9.58 seconds.* *Write this speed as:* *a) seconds per metre* *b) metres per second* *The men's world record for swimming 100 m (set in February 2024) is 46.80 seconds.* *Write this speed as:* *c) seconds per metre* *d) metres per second* *Compare the running and swimming speeds.* *e) Why do you think that metres per second is the more common compound measure used for speed?*	Compound measures are essentially functional multipliers since all compound measures compare a measure of one type with a measure of a different type. Students need to be familiar with named multiplicative comparisons such as speed, pressure or density. The units of a compound measure are the units of one measure, in relation to one standard unit of the other measure. The **variation** in the units used in *Example 3* draws attention to this. Part e encourages students to see the use of compound measures as comparisons, and so although both 'seconds per metre' and 'metres per second' are valid, it makes sense that the faster speed should have the greater value. Students' **language** can be used to develop this understanding. Encourage them to verbalise the units of the compound measure. Instead of mph, phrasing as 'miles per hour' or 'miles per one hour' and as 'the distance travelled for every hour that passes' to help reinforce the notion of per unit measure. Offer a double number line or a ratio table as a **representation** and use it to discuss what the different multipliers represent in terms of compound measures (for example, in a double number line the functional multiplier between the lines will be the rate of either ms^{-1} or sm^{-1}). Students meet compound measures in different subject areas. Is there agreement across your school about how best to work with compound measures? For example, formula triangles are sometimes used when working with compound measures such as speed and pressure. Are they used in your department or school? And, if so, what discussions might need to take place about forming a coherent approach that promotes understanding and connection?

Example 4:

The images below all represent an average speed.

The calculation $165 \div 3 = 55$ *is relevant to each representation.*

Look at each representation and decide:

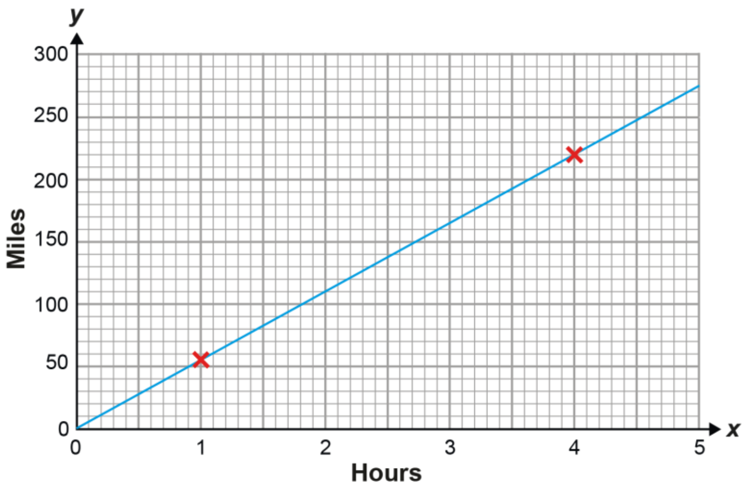
a) *Where are the 165 and the 3 in each representation?*

b) *What does the 55 mean in each representation? What does it mean in terms of the average speed?*

In *Example 4*, the calculation in the question is deliberately different to the pair of values drawn attention to in the **representations**. This is to encourage students to make connections to the multiplicative constant as the rate of change in the context of compound measure. The multiple representations here are also intended to make explicit the role that the calculation $x \div y$ plays, and to model the constant interpretation of that calculation in terms of the compound measure that it represents.

Consider **deepening** students' understanding by replacing the labels of miles and hours with, for example, Newtons and square metres, telling students that these diagrams now represent pressure and asking what the units of pressure might be.

Repeat this with kilograms, cubic metres and density; **variation** in units will draw attention to the constant nature of the relationship $x \div y$.



Hours	1	4
Miles	55	220

Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 8 Proportional reasoning* can be found at https://www.ncetm.org.uk/media/4gdbakqz/ncetm_ks4_cc_8_solutions.pdf