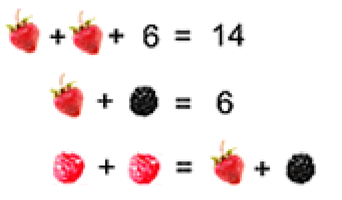
Secondary Magazine 116



National Centre for Excellence in the Teaching of Mathematics

The days are getting shorter, but that's no reason for maths gloom: you can ask your students to plot a graph of the times of sunrise and sunset and then try to predict the times for next week, or even to model the pattern they observe! This is the third edition of our refreshed Secondary magazine; let us know what you think, by email to info@ncetm.org.uk or on Twitter, @NCETMsecondary.

Contents

Heads Up
Here you will find a check-list of some of the recent, or still current, mathematical events featured in the news, by the media or on the internet: if you want a "heads up" on what to read, watch or do in the next couple of weeks or so, it's here. This month there's maths all over our screens (big and small), and we've included podcasts about Further Maths, short videos about women mathematicians, links to articles from ATM, and information about Bowland grants.

Building Bridges
The regular feature in which discussion of secondary mathematics topics draws out the inter-connectivity of the topics with preceding, succeeding or surrounding topics, in ways that will support and enrich your teaching in KS3 and KS4. This month: division of and by fractions.

Sixth Sense
Stimulate your thinking about teaching and learning A level Maths, with these monthly articles from Andy Tharratt (NCETM's level 3 specialist Assistant Director). This month he writes about teaching mechanics.

From the Library
Want to draw on maths research in your teaching but don't have time to hunker down in the library? Don't worry, we've hunkered for you. In this issue you can be inspired by an article considering pupils' understanding of algebra, *The 'algebra as object' analogy: a view from school.*

It Stands to Reason
Developing students' reasoning is a key aim of the new KS3 and 4 Programmes of Study, and this monthly feature shares ideas how to do so. In this issue we think about developing algebraic reasoning.

Eyes Down
A picture to give you an idea: "eyes down" for inspiration.

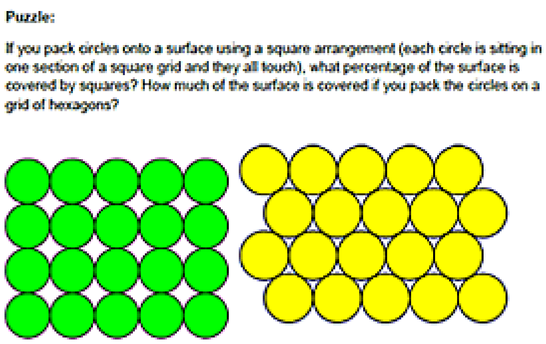
Heads Up

Whilst not wanting to define this article as mathematical gossip, it does come close! We've brought together news and current mathematical affairs, all in one place. We do hope it will interest you.

Maths is all over the mainstream at the moment! *The Imitation Game* (about Alan Turing) and *The Theory of Everything* (about Stephen Hawking) are in the multiplexes, and both ITV's Tonight and BBC1's *The One Show* have broadcast features about maths teaching – you should be able to find them on a catch-up service before they expire

Ofsted has published a document, Ofsted inspections - clarification for schools, providing clarification for schools. Whilst this clarification is not maths specific, you may find it useful, not just if you have an inspection imminent!

The Further Maths Support Programme (FMSP) is producing a series of podcasts entitled 'Taking Maths Further'. As an example, in Episode 8 Packing Shapes and graphene, the topic was the most efficient way to pack shapes in 2D and 3D space. There was an interview with Jacek Wychowianec, a scientist studying applications of materials science to biology. He talked about how he uses many different types of maths in his work, and how he's been developing substances which can be used to help regrow damaged nerves. Each episode has support materials which include some links and a puzzle



Puzzle:

If you pack circles onto a surface using a square arrangement (each circle is sitting in one section of a square grid and they all touch), what percentage of the surface is covered by squares? How much of the surface is covered if you pack the circles on a grid of hexagons?

Have a look at the Nottingham collection of short videos on women mathematicians, each about 90 seconds to three minutes long. They are particularly suitable for A-level students (and aspiring undergraduates). These link with the wider activity of the Your Life campaign, who are putting together lots of stimulating and high-quality resources to inspire study of Maths and Physics post-16. The campaign is particularly targeted at encouraging more girls to study these subjects.

The Bowland Trust is offering a second round of grants, this time for up to £1000. The main aim is to stimulate activities that extend the use and understanding of Bowland maths and its materials, but with more emphasis on encouraging new users. Bids should be made by **30 November**.

We are pleased to include links to two articles from the latest journal published by the Association of Teachers of Mathematics (ATM). In the first article, The Museum of Mathematics (MoMath) - the next Generation of Science Museum, Daniel Short and Allison Peters report on this new development in New York that seeks to foster interest in mathematics. In the second article, Mathematics for Parents, Tony Cotton explains how working with parents where English might not be their first language can support the learning of their children. He then describes 'what should have been a maths lesson'.

Image credit

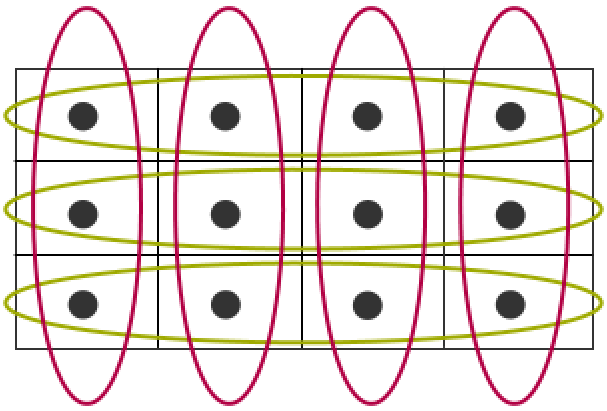
Building Bridges
Dividing fractions: how to share $7\frac{1}{5}$ apples between the proverbial $2\frac{2}{5}$ children

In Issue 115 we considered how to build bridges from pupils' understanding of addition, subtraction and multiplication in the context of integers to that of fractions. Let us now consider division, and then division by (non-integral) fractions.

Many pupils first meet division at primary school in the context of "equal-sharing" problems: "There are 3 children at a party and 12 sticky buns are shared equally between them. How many buns does each child get?" Glossing over the incongruity of this scenario – it's not much of a party if there are only 3 guests, and in the modern day of healthy eating those children will be lucky to get anything more than half an organic low-GI oat and quinoa muesli bar – there is also a conceptual weakness in the "equal-sharing" model for division: how does it extend to non-integer dividends, divisors and quotients? Is there a model that describes $12 \div 3 = 4$, $12 \div 5 = 2.4$ and also $12 \div 2.4 = 5$?

Secondary teachers, especially KS3 teachers, gain a lot from talking with their schools' feeder primaries, for example knowing which models for division their pupils have met. It is likely that your pupils will also have met the "equal-grouping" model: "$12 \div 3$" is represented by taking groups of 3 objects (apples, buns, muesli-bars) away from 12 of the objects, until none are left. Because this can be done 4 times, we say that $12 \div 3 = 4$. There's a conceptual link that can be made here between division being thought of as repeated subtraction ($12 – 3 – 3 – 3 – 3 = 0$) and multiplication (the inverse of division) being thought of as repeated addition (the inverse of subtraction).

Thinking of the division $12 \div 3$ as an activity, the "equal-sharing" algorithm is "give a bun to child A, a bun to child B, a bun to child C, then give another bun to child A, another bun to child B, another bun to child C, and so on"; the "equal-grouping" algorithm is "take 3 of the buns to make group A, then take 3 to make group B, then another 3 to make group C, and so on". The "equal-sharing" model of $12 \div 3 = 4$ can be verbalised as "12 divided BETWEEN 3 (recipients)", whereas the "equal-grouping" model can be verbalised as "12 divided INTO 3's". Thinking of an array of 3 rows by 4 columns as a representation of $3 \times 4 = 12$, the "equal-sharing" model of division tells us that there are 4 dots in each of the 3 rows, and the "equal-grouping" model tells us that there are 3 dots in each of the 4 columns:



A pupil can use the "equal-grouping" model to describe $12 \div 2.4 = 5$ (assuming the objects are, like apples / buns / muesli-bars, divisible into smaller pieces, so that you can put $2\frac{2}{5}$ objects in each group), but it's harder to use it to model $12 \div 5 = 2.4$: the pupil has to work out what to do with the last 2 objects, having first taken away 5 and then another 5.

A third, and we would argue more powerful, model of division is to think of $12 \div 3 = 4$ as of fitting 3's into 12, and being able to fit 4: for example, the activity of fitting 4 equal sticks each of length 3cm into a gap of 12cm:

The power of this "equal-fitting" model is that it extends naturally to non-integer dividends, divisors and quotients, and also that it bridges back to very early numeracy work. Consider how manipulatives such as Cuisenaire rods would help all pupils, especially hitherto low-attainers, have secure access to examples such as the following, and from there be able to reason about the underlying concepts.

- We model $12 \div 2\frac{2}{5}$ as fitting sticks of length $2\frac{2}{5}$cm into a gap of 12cm. We use exactly 5 sticks to do so. We interpret this as $12 \div 2\frac{2}{5} = 5$

- We model $12 \div 5$ as fitting sticks of length 5cm into a gap of 12cm. When we do so, we use 2 sticks and then need to use a bit – a fraction – of a 5cm stick to fit into the remaining 2cm gap: we need $\frac{2}{5}$ of a stick. We interpret this as $12 \div 5 = 2\frac{2}{5}$

- We model $12 \div 1\frac{1}{2}$ as fitting sticks of length $1\frac{1}{2}$cm into a gap of 12cm. To do this, we can start with 4 sticks of length 3cm fitting into the gap, and then halve each stick into two pieces each of length $1\frac{1}{2}$cm. This will give us twice as many sticks, so that now 8 sticks fit the gap. We interpret this as $12 \div 1\frac{1}{2} = 8$

- We model $12 \div \frac{3}{10}$ as fitting sticks of length $\frac{3}{10}$ cm into a gap of length 12cm. To do this, we can start with 4 sticks of length 3cm fitting into the gap, and then chop each stick into ten pieces each of length $\frac{3}{10}$ cm. This will give us ten times as many sticks, so that now 40 sticks fit the gap. We interpret this as $12 \div \frac{3}{10} = 40$ … I won't draw the picture!
- To determine $120 \div \frac{3}{10}$, we can start with 4 sticks each of length 3cm fitting into a gap of 12cm. When the gap becomes ten times bigger, then we need ten times more sticks, which we interpret as $120 \div 3 = 40$. Now each stick is chopped into ten pieces each of length $\frac{3}{10}$ cm, so that now we have ten times as many sticks. Therefore 400 sticks fit the gap. We interpret this as $120 \div \frac{3}{10} = 400$.
- Clearly, $12 \div 3$ will equal the same as $24 \div 6$: making both the gap and the sticks twice as big won't change the number of sticks needed. The step from here to having a secure conceptual understanding of equivalent fractions is one that most pupils will be able to make and explain.

The "sticks and gaps" model gives pupils the language to talk about what they observe when considering scale factor changes to the dividend and divisor in a calculation $A \div B$:

- if the dividend (the numerator) A increases by multiplication by a scale factor then this can be modelled as the gap getting bigger, and so the answer to the original division, the original quotient $A \div B$, which we model as the number of sticks we originally needed, gets bigger by the same scale factor i.e. $AC \div B \equiv (A \div B) \times C$;

- if the divisor (the denominator) B is multiplied by a scale factor > 1 then this can be modelled as sticks being glued together to make bigger sticks, and so the original quotient, the number of sticks we originally needed, gets smaller by the same scale factor i.e. $A \div BC \equiv (A \div B) \div C$;

- if the denominator B is divided by a scale factor > 1 then this can be modelled as the sticks being broken into smaller sticks, and so the original quotient gets bigger by the same scale factor i.e. $A \div (B \div C) \equiv (A \div B) \times C$;

- if the numerator A is divided by a scale factor > 1 then this can be modelled as the gap getting smaller, and so the original quotient gets smaller by the same scale factor i.e. $(A \div C) \div B \equiv (A \div B) \div C$.

Putting this altogether, pupils develop deep conceptual understanding of the algorithm for dividing a number N (which can be an integer or a non-integral decimal or fraction) by a non-integral fraction $\frac{c}{d}$: they understand that making the sticks of length c smaller by a factor of d (i.e. breaking each stick into d equal pieces) means that d times as many of them are needed compared to the original amount needed (which would be $N \div c$). Put formally:

$$N \div {}^{c}/_{d} \equiv N \div (c \div d) \equiv (N \div c) \times d \equiv N \div c \times d \equiv N \times d \div c \equiv N \times {}^{d}/_{c}$$

though pupils need to be confident with the order of operations to follow, or create, this argument.

Once pupils understand why the denominator d of the divisor has a multiplicative effect on the dividend, procedural fluency is much more likely to be developed and embedded.

Image credit

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National Centre for Excellence in the Teaching of Mathematics

Sixth Sense

Teaching Mechanics through modelling, mistakes and misconceptions

Introducing Newtonian Mechanics is one of the greatest pleasures of teaching level 3 Maths, and it is also one of the greatest responsibilities: it's so important when choosing the first examples and scenarios for your students to consider that the right thought-trains (pulling wagons against negligible resistance with light inextensible couplings between them, of course!) are set in motion. A seemingly simple question – "why do you fall backwards when the bus lurches forward?", "if action and reaction are equal and opposite, why does something move when I push it?" – can generate deep and enlightening discussions that open students' eyes to the power and the elegance of Newton's insight. If only there were a more modern and appealing name for this area of applied mathematics than Mechanics! This is a word that seems to put students off rather than draw them in: so how about "Modelling Reality" instead? There are many web resources that can be used to inspire students' interest: a particular favourite is Assembler, which like all the best games is so simple, so frustrating, so instructive and so addictive! The NRICH site has the rich sub-site Mathematical Models in Mechanics, with lots of puzzles, challenges and problems that will help your students develop and refine their mechanical thinking.

The excellent National STEM Centre eLibrary is available – for free! – to all teachers. It includes some splendid resources both old and new for teachers of mechanics in level 3 Maths. A quick search in the eLibrary using the key word "mechanics" comes up with lots of practical approaches to developing modelling skills, and to challenging the common misconceptions that often occur when students first study Newtonian mechanics.

Mechanics in Action by Savage and Williams (CUP 1990) and *The Teacher's Guide to the Leeds Mechanics Kit* by Jagger, Roper and Savage (MAP 1990) give lots of practical ideas for conceptual understanding across mechanics topics, from introductory vector quantities in AS Level Mathematics to forced and damped harmonic motion in A Level Further Mathematics. The rich ideas here are still valid, even though some teachers may themselves have first met these when they were in the sixth form themselves – fortunately, Newton's Laws have not changed in the last 25 years! Old textbooks advocating similar approaches are also available in the eLibrary: try *Nuffield Advanced Mathematics: Mechanics 1* and *Mechanics 2* by Hugh Neill (Longman 1994). For more motivational content, try *Exploring Mechanics* from the Centre for Teaching Mathematics (1995) or *Realistic Applications Of Mechanics* from the SPODE Group (OUP 1986).

If you already use all of the above and want something new, then content such as the *Bloodhound SSC Secondary Topics* from 2008 to present and *Integrating Mathematical Problem Solving: Physics – Simple Harmonic Motion* (MEI 2012) are up to the minute. They make considerable use of modern ICT such as Flash demos … but do check that your school firewall will let you access these sorts of resources before planning an hour's lesson that hinges on them!

Image credit

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From the Library
Shh! No Talking!

Our regular feature highlighting an article or research paper that will, we hope, have a helpful bearing on your teaching of mathematics

In this issue, It Stands to Reason focusses on pupils reasoning in algebra, so our library article this month is The ‘algebra as object’ analogy: a view from school by Kate Colloff and Geoff Tennant. This paper discusses the ‘algebra as object’ analogy (eg "saying that ‘a’ means ‘apple’") and finds out how pupils would explain a simplification like 3a+2a = 5a.

The authors report that:

> *"When asked directly about the use of the ‘algebra as object’ analogy, all teachers were aware that it was ‘not recommended’ although one teacher could not articulate why. Reasons given by others included, “You cannot multiply apples”."*

And they highlight:

> *"the need to problematise methods of teaching simplification which do not use ‘algebra as object’: from a learning point of view, is it all that different, for example, to refer to “2 lots of 10 plus 3 lots of 10” rather than apples?"*

Is this paper relevant to your current teaching? This could be the stimulus for a discussion in your department about how you work with pupils to overcome difficulties understanding symbols

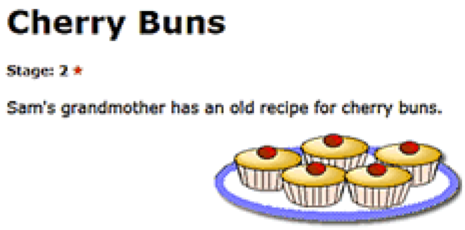
We think that you will, having read this paper, have a deeper understanding of why some pupils find algebra difficult, and also have some ideas how to respond to this in order to help your pupils develop more secure conceptual understanding and procedural fluency. Let us know what you think.

Image credit

It Stands to Reason

In this regular feature, an element of the mathematics curriculum is chosen and we collate for you some teaching ideas and resources that we think will help your pupils develop their reasoning skills. If you'd like to suggest a future topic, please do so to info@ncetm.org.uk or @NCETMsecondary.

In Issue 114, this article focused on sequences. In this issue we will consider other algebraic contexts in which pupils can work with to develop their reasoning skills before solving formal equations, which we will cover in a future issue.



To provide some motivation to think algebraically, the NRICH problem Cherry Buns uses the context of a recipe (you may remember older relatives using a recipe like this, pre-Delia!) to reason with numbers. Although there is no explicit use of symbols, pupils may instinctively use a symbol or picture to represent the weight of an egg.

Pupils are guided into using some sort of algebraic reasoning is the resource Is it Magic or Is it Maths?, again from the NRICH website. This gives some starting points that encourage the use of algebraic thinking to solve problems

There are many examples of using a 'missing number' problem. This can be exemplified in the Autumnal Relationship resources.

The slides in this sequence get progressively more difficult without ever using an *x* or *y*: pupils can begin to feel comfortable in reasoning with symbols of a different kind before letters are introduced.

To develop further pupils' confidence with the language we use when talking about symbols in maths lessons, you could use a set of expressions such as:

n	$n+2$	$n-2$	$2n$	n^2

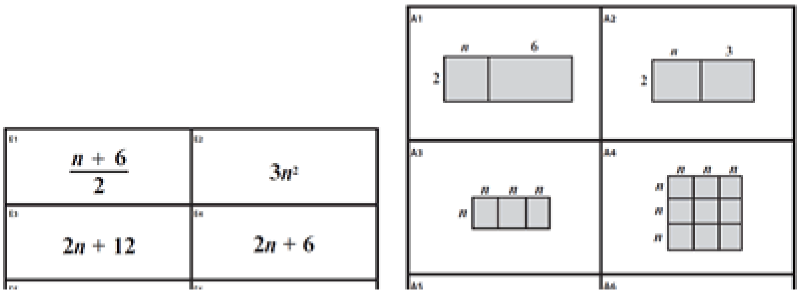
These can be displayed on cards that pupils manipulate either on their desks or as a demonstration in front of the class. You would ask a pupil to put the cards in order of size, prompting the realisation that a value of n needs to be given before the task can be completed unambiguously. Then you would ask pupils to suggest values, and the cards would be ordered in the different cases.

Further questions could be:

- Can you find a value of n so that the value of n^2 is the biggest of all the values?
- Is the value of n–2 always smaller than the value of n?
- Is the value of 2n always bigger than the value of n?
- Can you find a value of n so that the values of n+2 and 2n are the same?

Note that it is essential that the pupils say that they are comparing the **value** of n–2 with the **value** of n, and not that they are comparing n–2 with n: expressions aren't numbers, so one expression can't be bigger than another.

While the Improving Learning in Mathematics resources were written originally for post–16 learners, the activity [Interpreting algebraic expressions] (available from the National STEM Centre eLibrary) will give pupils further opportunities to reason with the meaning of expressions.



The lesson plan attached to the resource gives some other useful suggestions.

Here are further suggestions:

- Anne Watson's article What's x Got to Do with It? discusses algebra in the new national curriculum
- This algebra mystery
- The SMILE booklet Algebra Makes Sense includes many good activities.

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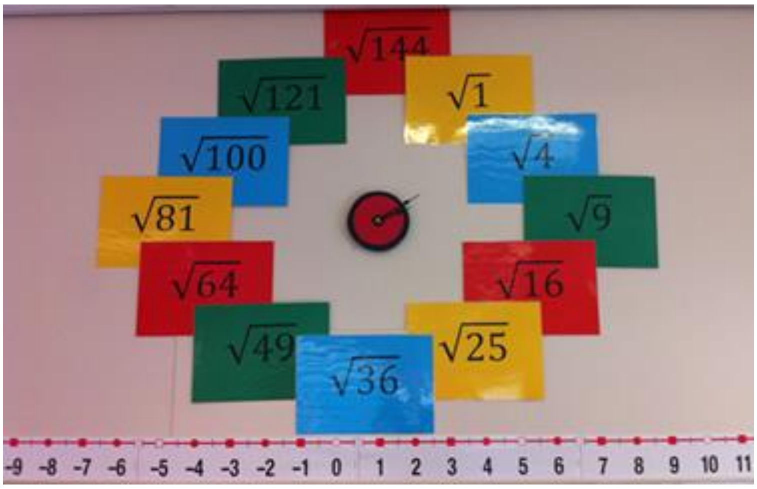
Eyes Down

Our monthly picture that you could use with your pupils, or your department, or just by yourself, to make you think about something in a different way



This clock is a definite talking point! You could ask your pupils to:

- explain the symbols around the clock face;
- describe the patterns that the numbers on the same-coloured paper seem to have;
- design their own clock face with numbers displayed in a "coded" way: KS3 pupils will like the challenge to do using only four 4's and the basic arithmetic operations (so $1 = 4 \div 4 + 4 - 4$, etc.), while older pupils could use more complicated expressions (e.g. 0.2^{-1} or $2^{\log_3 27}$).

If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb) to us at info@ncetm.org.uk, with a note of where and when it was taken, and any comments on it you may have.

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