Secondary Magazine 131

National Centre for Excellence in the Teaching of Mathematics

Welcome to Issue 131 of the Secondary and FE Magazine

We all have to face the "why do we need to know this?" question from time to time, but if it is posed in a statistics or data handling lesson, there is, certainly for the next few months, a very strong and robust response: because at the moment data, information and statistics are being quoted, analysed, manipulated and misused in and by politicians and the media, to an extent that feels unprecedented. The EU referendum, the US presidential race, the conflict in the Middle East: the debate in these and so many other areas is thick with statistical claims and counterclaims, and everyone needs the knowledge and the tools to make headway through the data haze, to get closer to the truth that almost certainly isn't what we are reading or hearing. As Maths teachers we have the opportunity to help our pupils, and through them their families, hear the sense despite the noise: indeed, it could be argued that we have a duty to do so. Let us know, by email to info@ncetm.org.uk or on Twitter @NCETM. If, over the coming weeks, you have any examples of successfully helping your pupils understand the complexity behind the soundbite, please do share them with us.

Contents

Heads Up
Here you will find a checklist of some of the recent, or still current, mathematical events featured in the news, by the media or on the internet: if you want a "heads up" on what to read, watch or do in the next couple of weeks or so, it's here. If you ever think that our heads haven't been up high enough and we seem to have missed something that's coming soon, do let us know: email info@ncetm.org.uk, or via Twitter, @NCETM.

Building Bridges
A number of conceptually challenging ideas that pupils meet in KS3 rely directly thought not always obviously on their understanding of KS2 arithmetic. In this issue, we explore what these foundations are, and how we can try to ensure their strength and solidity.

Sixth Sense
Calculator use with GCSE re-sit students in FE colleges: given the pressure in re-sit classes, is it worth giving time to exploring and developing calculator skills? Yes, definitely, we argue

From the Library
An introduction to "Realistic Maths Education", which was developed in the Netherlands, in part in response to the "why do we need to know this?" question.

It Stands to Reason
We consider John Mason's imaginative less-same-more grids, and explore in depth one example to show how rich and deep challenge can arise from the simplest of contexts.

Eyes Down
A picture to give you an idea: there can be too much of a good thing.

Image credit

Heads Up

The NCETM's Director, Charlie Stripp, has been giving his views on a wide range of maths education topics in a podcast on the Mr Barton Maths website. Of particular interest to secondary teachers are sections on: technology (at 5:00); the new A level (15:00); the new GCSE (39:15) and teaching for mastery (58:00).

What subject knowledge does a secondary maths teacher need? Is an A level in the subject enough? Or a degree? The answer is that neither is enough. Read more in the latest issue of *Bespoke*, the magazine of the Maths Hubs programme.

Maths teachers and PhD students are wanted, to help the qualifications watchdog, Ofqual, conduct a research study into the level of difficulty of sample questions for the new A level maths. Teaching for the new A level is due to start in autumn next year (2017). Comparisons between different exam boards sample questions will be made in July this year.

Who says maths is dry? We defy anyone confronted by a graphic explanation of the golden ratio not to be awestruck by its neatness and beauty. Maybe that explains why it found a place in the BBC Radio Four series, A History of Ideas. It's an episode that still receives numerous references on social media.

Finally, an update on our item last month reporting that London maths teacher Colin Hegarty, was named in the top ten finalists for a global teaching prize. Well, he didn't scoop the million dollars. First prize went to a teacher in a Palestinian refugee camp.

Image credit
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Building Bridges

Three discussions we had in a recent department meeting began independently but soon became closely connected: a colleague asked for some suggestions about teaching his Year 7 class to simplify expression with products of algebraic terms; a different colleague was wondering why his Year 9 group of high prior attainers were finding multiplying and dividing numbers written in standard form very hard to grasp; and a third colleague's Year 8 class were struggling to calculate areas of triangles. We explored the mathematics underpinning these seemingly different topics, and agreed that each assumed – required, even – confident understanding of, and fluency with, manipulating what we called "calculation strings", in particular strings of multiplications and divisions. The Year 7 pupils needed to be confident that $3 \times 4 \times 5 \times 6 = 3 \times 5 \times 4 \times 6$ before they could simplify with understanding $3 \times a \times 4 \times b$; the Year 8 pupils needed to be sure that $0.5 \times (6 \times 14)$ would be the same as 3×14 or 6×7 but not the same as 3×7 – that they shouldn't be "multiplying out the brackets"; and if the Year 9 pupils were to simplify the product and the quotient of 4×10^9 and 8×10^3, first they needed to simplify securely the product and the quotient of 4×9 and 8×6. How then, we discussed, should the depth and strength of the prerequisite knowledge be assessed – and then consolidated, if found to be lacking?

We knew that our pupils would have first experienced manipulating and simplifying calculation strings in Key Stage 2, and should have linked abstract statements such as the commutative law of multiplication $a \times b = b \times a$ to concrete representations such as the area of two rectangles, each with the same dimensions but one rotated a quarter turn relative to the other. They might have seen volume of a cuboid as a good representation of the associative law of multiplication: the "base" area can be any of $a \times b$, $b \times c$ or $c \times a$, so the volume can be any of $(a \times b) \times c$, $a \times (b \times c)$ or $(c \times a) \times b$. We had to build the bridge from there, and so we planned together activities such as

Agree or Challenge?

$3 \times 4 \times 25 = 300$

$19 \times 8 \times 125 = 19000$

$23 \times \frac{1}{4} \times 4 = 23$

$31 \times \frac{1}{5} \times 15 = 93$

Copy and complete …

$17 \times 4 \times 25 = \square$

$125 \times \square \times \square = 12000$

$50 \times \frac{1}{6} \times \square = 200$

$50 \times \square \times 6 = 200$

$50 \times \frac{1}{3} \times \square \times \square \times 8 = 3000$

not only to assess our pupils' familiarity with what is and isn't permissible when manipulating multiplication strings but also to give them motivation for the formal laws of associativity and commutativity: we wanted them to realise how vexing it would be if $3 \times 4 \times 25$ didn't equal $3 \times (4 \times 25)$! We were confident that once the pupils wanted to be sure that they could "start the calculating with the nice products", then exploring

What's the same, what's different?

$2 \times 3 \times 4 \times 5$

$2 \times 3 \times (4 \times 5)$

$(2 \times 3) \times 4 \times 5$

$2 \times (3 \times 4) \times 5$

What's the same, what's different?

$2 \times 3 \times 4 \times 5$

$2 \times (3 \times (4 \times 5))$

$2 \times 3 \times (4 \times 5)$

$2 \times (3 \times 4) \times 5$

$(2 \times 3) \times (4 \times 5)$

$((2 \times 3) \times 4) \times 5$

would seem to them worthwhile and relevant. We agreed in the department that we would use, and we would ensure that the pupils used, precise language: as one colleague said, if his son could reel off – and spell! – the Latin names of 20-odd dinosaurs in Year 3, then in Year 7 he could certainly be expected to remember and use “commutative” and “associative” when simplifying multiplication strings

Agree or Challenge?

$25 \times 76 \times 4$

$\equiv 76 \times 25 \times 4$ Commutative Law

$\equiv 76 \times (25 \times 4)$ Associative Law

$\equiv 76 \times (100) =$ 7600

Work out in the same way

$4 \times 47 \times 25 \equiv \ldots = \square$

$8 \times \square \times 125 \equiv \ldots = 5000$

$\frac{1}{6} \times 85 \times \square \equiv \ldots = 85$

$\square \times 50 \times 6 \equiv \ldots = 150$

$\square \times 40 \times \frac{2}{3} \times \square \times \frac{1}{4} \equiv \ldots = 300$

and when reasoning

True or False?

A. $9 \times 11 \times 13 = 11 \times 9 \times 13$ because of the associative law of multiplication.
B. $9 \times 11 \times 13 = 9 \times (13 \times 11)$ because of the commutative law of multiplication.
C. $7 \times 8 \times 25 = 7 \times 2 \times 4 \times 25$ because of the associative law of multiplication.
D. $56 \times 25 = 14 \times 100$

Odd One Out

A. $7 \times 8 \times 5 \times 9 = 8 \times 7 \times 5 \times 9$
B. $7 \times 8 \times 9 \times 5 = 7 \times (8 \times 9) \times 5$
C. $(5 \times 7) \times 8 \times 9 = (7 \times 5) \times 8 \times 9$
D. $(5 \times 7) \times (8 \times 9) = (7 \times 5) \times (9 \times 8)$
E. $(5 \times 7) \times (8 \times 9) = (9 \times 8) \times (7 \times 5)$

We decided to develop our pupils’ understanding of and fluency with mixed strings similarly: beginning with “3 numbers, 2 operations”

What’s the same, what’s different?

$6 \times 9 \div 3$

$(6 \div 3) \times 9$

$9 \times (6 \div 3)$

$6 \times (9 \div 3)$

Complete …

$42 \times 12 \div 4 = \square$

$51 \times 8 \div 17 = \square$

$\square \times \frac{1}{3} \div 4 = 5$

$50 \times 7 \div \square = 14$

$\square \div 3 \times \square = 4$

$\square \div \square \times \frac{3}{5} = 2$

and then going deeper with

What’s the same, what’s different?

$60 \div 5 \div 4$

$(60 \div 5) \div 4$

$60 \div (5 \times 4)$

$60 \div 4 \div 5$

Complete …

$350 \div 7 \div 5 = \square$

$55 \div 1\frac{2}{3} \div 3 = \square$

$\square \div 4 \div 5 = 7$

$72 \div 4 \div \square = 36$

$\square \div 6 \div \frac{1}{3} = 11$

$\square \div \square \div \frac{3}{5} = 6$

and developing their reasoning with

True or False?

1. 13682 ÷ 563 × 71 = 13682 ÷ 71 × 563
2. 56428 × 923 ÷ 413 = 923 ÷ 413 × 56428
3. 78408 ÷ 2 ÷ 9 = 78408 ÷ 3 ÷ 6
4. 3600 ÷ 17 ÷ 9 = 2000 ÷ 17 ÷ 5
5. 455 × 122 = 91 × 610 because of the associative law of multiplication.

We agreed not to progress immediately to “4 numbers, 3 operations” – i.e. (4 × 9) ÷ (8 × 6) is the same as (4 ÷ 8) × (9 ÷ 6) and also (4 ÷ 6) × (9 ÷ 8) – so that all the new ideas, thinking and language around “3 numbers, 2 operations” would have time to settle and be consolidated … and we rejigged all the schemes of work to postpone the topics that had prompted this discussion in the first place!

This serendipitous discussion (much more worthwhile than the admin that was on the meeting agenda!) reminded us starkly how important it is that a scheme of work is written AFTER a detailed concept map or timeline has first been written: the pupils in the Year 8 and 9 classes were struggling, and those in the Year 7 class were likely to, because the sequence of **contexts** wasn’t in step with the sequence of **concepts**. We were hoping for procedural fluency, but we hadn’t ensured first their factual knowledge or their conceptual understanding: without doing so, we were almost certainly hoping in vain.

You can find previous *Building Bridges* features here.

Image credit

Sixth Sense

GCSE resit students in the Post-16 sector know that the PASS is all important, and that a C on their results slip will mean that more, and more rewarding, doors are open to them. But they have many barriers to learning that they have built around them over the previous eleven (at least) years of studying maths and, as they see it, failing it again and again. GCSE resit tutors in Post-16 colleges have the task of building confidence as well as teaching maths, of breaking down those barriers and showing their students that it's not that they have failed in the past, it's that they just haven't passed YET.

The skills of calculator use are frequently brushed over in the Post-16 classroom. It's assumed that the students have been using their devices all through school, and that in the modern digital age every student knows exactly which button to press to get the right answer. Unfortunately, more often than not, this is most certainly not the case.

It is common for students not to own a calculator. The version that they had was lost or disposed of in that magic time between the exam in Y11 and results day, when they hoped beyond hope that they had passed. Why would you walk around with a scientific calculator to hand? To be honest, in everyday life, it's fine to whip out your phone and use its calculator, or – more likely – an app or a search engine to calculate for you. And when students do come to lessons equipped with a calculator, you are often faced with a multitude of makes, the legacy of each of your college's many feeder schools having their own "recommended" model … and not one instruction manual!

With 50% of marks in the current GCSE being on a calculator paper, it's vital that students have access to a calculator and that they know how to use it effectively and efficiently in order to gain as many marks as possible. Once the 9-1 Maths GCSE kicks in in FE, this could rise to two-thirds of marks being allocated to calculator papers. Sometimes, students are so eager to show as much "actual maths" as possible, they attempt by hand questions on the calculator paper which could have been done a lot more efficiently with the pressing of a few buttons.

So, bearing all of this in mind, here's a short-and-by-no-means-exhaustive set of strategies and ideas to ensure that you are helping your resit students make the most of their new best maths-friends: their calculators.

Write down intermediate steps

There are questions which specifically test calculator entry, and mistakes can be made by pressing a 1 instead of a 2. Students should always work through the problem step by step and record as they go by showing results from their calculator screen, not just the final solution.

Use calculators as instant feedback machines

Students can be encouraged by being able to see that they are getting the right answers to mental and/or written calculations. Calculators don't criticise or express disapproval at wrong answers – they can actually motivate students to try again.

Back things up

Practise skills of estimation and approximation alongside calculator skills. If students have a rough idea of the answer they are looking for, they are more likely to examine the solution the calculator provides and check it makes sense. And this is an important workplace (and life-in-general) skill, much valued by employers.

Let students play

Students need to get used to how their calculators work. Set aside time for them to play with their devices. Give them freedom to figure out what buttons to press and when – this is very like how we first learn computer games or how to work apps. Quick tutorial, then have a go!

Compare and contrast

It can be a useful exercise to give students the same calculation, and explore the key press sequences on different models of calculator. This can bring out lovely mathematical discussion, especially with regards to order of operations. Using a visualiser or similar can show students what buttons to press on the big screen - for example, when squaring a negative number, some models will put in brackets for you, whilst others expect you to add these yourself.

No "No phone Zone"

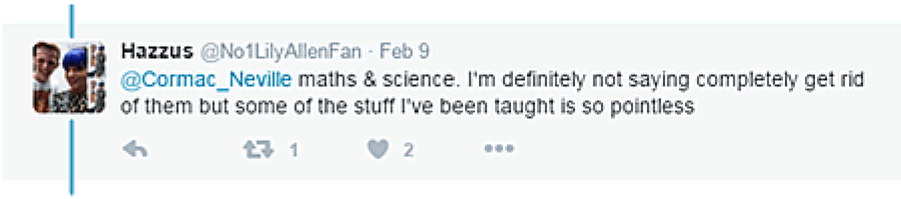
Why? Let them get their phones out. All mobile phones have a calculator function, and it's important that students learn how to use it. Granted, they cannot use it in an exam, but for life outside studies, knowing the intricacies of the first genuinely universal pocket calculator is vital – especially since many phone calculators don't follow the operation rules that are programmed into scientific calculators, and so will declare that $3 + 2 \times 5 = 25$ not 13.

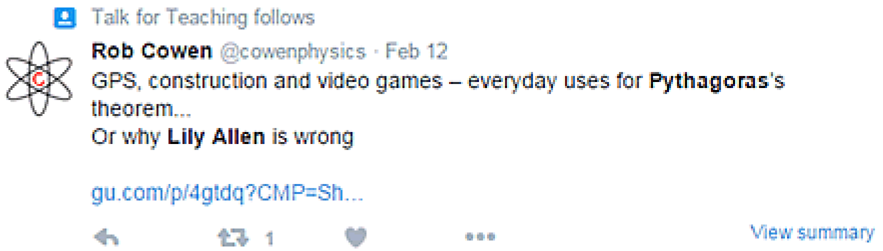
What have you done to overcome your students' uncertainty with, perhaps even anxiety about, their calculators? What activities, challenges and games have you found to be effective? Let us know, by email to info@ncetm.org.uk, or via Twitter, @NCETM.

You can find previous *Sixth Sense* features here.

From the Library

Lily Allen’s recent tweet to Schools’ Minister Nick Gibb – “I left school 15 years ago and I've not used Pythagoras's Theorem once” – generated a wave of responses. Some agreed with her sentiment, citing the irrelevance of their school mathematics experiences; some justified the place of school mathematics by the critical role mathematics plays in today’s ubiquitous technology:





Many of us in the classroom have to field questions along the lines of “But when am I ever going to use this piece of algebra / trigonometry / geometry / nonsense outside school?” One response to the challenge of making mathematics in the classroom relevant to all learners is Realistic Mathematics Education (RME), initiated in the Netherlands and subsequently explored in projects in the US and the UK.

The starting point for RME is that “students should develop their mathematical understanding by working from contexts that make sense to them” (Dickinson and Hough, 2012). Learners use their intuition to analyse a meaningful situation, whether from everyday life or even something purely mathematical. The key is that the context is “realistic”, in the sense of something learners can “realise”, from the Dutch verb “zich realiseren”, meaning “to imagine”. From their intuitive models, through guided instruction, learners are led to build more formal models, moving from “models for the situation” to “models of the situation” (van den Heuvel-Panhuizen, 2003, pp.14-15). An initial model may be as simple as a picture, but then this model is gradually refined and made more abstract, so that the model can become a tool for solving other problems. In this way models bridge the gap between the informal and the formal – but the route back to the context is always reviewed, so that the connection with the abstract is maintained. The development of more formal models is described as “progressive mathematisation” (Hough and Gough, 2007). Treffers distinguished between two types of mathematisation: “horizontal mathematisation”, the process of using “mathematical tools to organize and solve problems situated in real-life situations” and “vertical mathematisation” which “concerns moving within the abstract world of symbols”, “using connections between concepts and strategies” (van den Heuvel-Panhuizen and Drijvers, 2014, p. 522). The two modes of mathematising are considered of equal value and may overlap.

Dickinson and Eade (2005) give an example of the “progressive formalisation” of models in learning about fractions when investigating a problem concerning cutting up submarine sandwiches (figure 1). Models may start as pictures and then progress through levels of abstraction. The “term ‘model’ is not taken in a very literal way. Materials, visual sketches, paradigmatic situations, schemes, diagrams, and even symbols can serve as models” (van den Heuvel-Panhuizen, 2003, p.13):

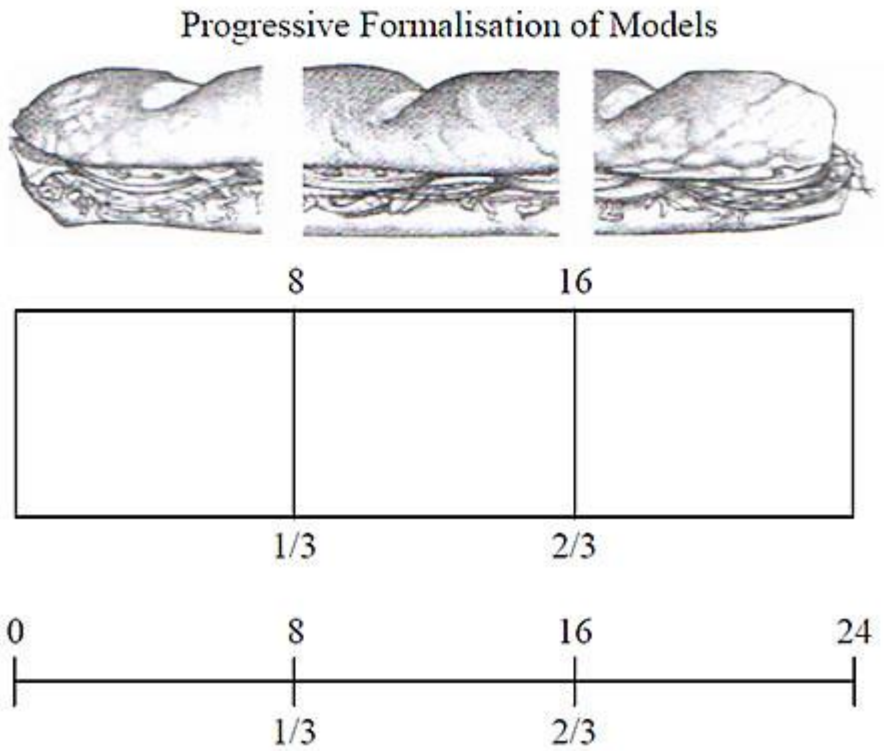


Figure 1 (Dickinson and Eade 2005)

According to Dickinson and Hough (2012), the distinctiveness of RME lies in:

- "Use of realistic situations" to allow students to develop their mathematics "as opposed to using contexts" for applying learnt mathematics.
- "Less emphasis on algorithms and more on making sense and gradual refinement of informal procedures."
- "Emphasis on refining and systemising understanding."
- "Less emphasis on linking single lessons to direct content acquisition and more on gradual development over a longer period of time. Students stay with a topic for long periods of time, remaining in context throughout."
- "Discussion and reflection play a significant part in supporting student development."

In an evaluation of two projects run in the UK, targeted at low prior attaining Key Stage 3 and Key Stage 4 pupils, the Centre for Evaluation & Monitoring at Durham University (Searle and Barmby, 2012) found a number of benefits as well as issues. The teachers "reported that the contexts and related activities interest the pupils and so engage them in the lesson … The starting contexts are rich and sometimes [the] pupils do not realise they are doing maths; this can only be a good thing for pupils who have so little confidence in the subject" (Dickinson and Hough, 2012)." It was found that several lessons might be needed to internalise the models, but once achieved the pupils "can understand how these models can be applied in a variety of contexts" (Searle and Barmby, 2012). "Exposing students to new, rich contexts and at the same time highlighting the 'mathematical' elements of these situations, allows children to learn maths that they see as 'relevant' but that also contains all the 'abstract' content that they would learn in a more 'traditional' classroom setting." (Dickinson and Hough, 2012). Using assessment data from Year 7 pupils it was found that "those pupils who had experienced RME were not only more likely to solve a problem correctly, but showed considerably more understanding through their ability to explain their strategy" (Searle and Barmby, 2012).

Key issues that emerged from the evaluation included concern from parents and school management that little was written in the pupils' exercise books, and that there was a lack of formal assessment. At the time of the evaluation there was also a perceived incompatibility with GCSE, although the problem solving approach may now be more in sympathy with the reformed examinations and the new (from September 2014) National Curriculum. The two other key issues identified by the evaluation related to pupils experiencing a mix of approaches, some teachers using RME, others not, and the need for "a support network of teachers" for "initial training and ongoing professional development" (Searle and Barmby, 2012). Teachers emphasised the need to understand the philosophy and to be trained in the use of the materials: "you can't just pick up the books and use them; it will not be effective" (Dickinson et al, 2011).

Perhaps this does not directly address Lily Allen's concern that the mathematics we teach is not explicitly relevant to most peoples' everyday lives. But if we believe a mathematical education is of value, does RME provide a route for making "mathematising" relevant and engaging? How do you respond to pupils' groans that maths is not relevant – or do you have an approach that overcomes their concerns from the outset? We would like to know your experiences.

Resources

MEI has a webpage devoted to RME including links to videos demonstrating use of RME resources.

The Freudenthal Institute has a number of applets to support "Maths in context".

Two of the references below, Hough & Gough and van den Heuvel-Panhuizen, contain examples of RME models and how they can be used.

The American Mathematical Society has produced Mathematical Moments, a series of posters and associated podcasts on a wide range of applications of mathematics, from thwarting poaching of rhinos, to designing rollercoasters, and improving understanding of the dynamics of cities.

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You can find previous *From the Library* features here.

Secondary Magazine 131

National Centre for Excellence in the Teaching of Mathematics

It Stands to Reason

> *"We need to do much more work with the most basic material to ensure that pupils grasp the relevant concepts. The last thing our more able pupils need is to be accelerated … able pupils may need challenges that are surprisingly basic, before they are confronted with material that is rich and sophisticated."*

Teaching Mathematics at Secondary Level, page 20, Tony Gardiner, 2014

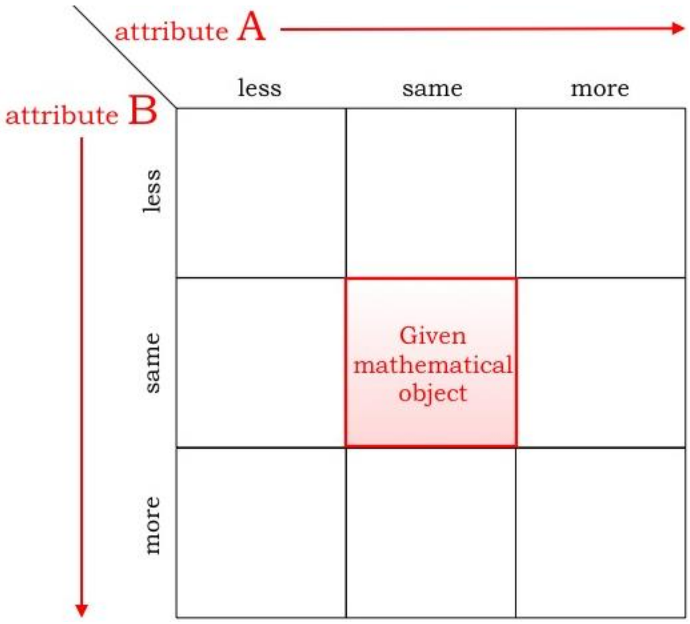
The Advisory Committee on Mathematics Education (ACME) offers similar advice; one of ACME's principles for mathematics education is that

> *"[all young people] experience mathematics in a deep, rich and connected way rather than being accelerated through a fragmented, test-driven curriculum."*

A blueprint for mathematics education: maths snapshots, Issue 1, ACME, June 2014

What kinds of task can we give pupils that will challenge even the highest achievers and the most "rapid graspers" (to use the Ofsted term), and that will develop their reasoning skills while involving only 'age-appropriate' (i.e. not accelerated or pulled forward) mathematical knowledge and concepts? In this article we look at a particular two-way-grid-of-cells that can be used to stimulate pupils' deep thought about any pair of related quantifiable concepts which can be applied to the same mathematical object – and the objects and concepts need not stray outside the expected content for the given pupils' age / year group.

We will look at a specific example shortly. The generic grid is as shown:



Idea from John Mason, *More or Less Perimeter and Area*, 2015

A task in which pupils work on an object represented in a less-same-more grid will provide excellent opportunities for them to reason about, and deepen their understanding of, mathematical concepts that are exemplified in the object – in particular, how they are related and how they interact. In so doing, pupils will usually "try to make sense of what has happened, what structure has been revealed, what inner aspects have been noticed while working on the task". (Ref: *Studies in Algebraic Thinking No 1*, *Inner and Outer Aspects of Tasks*, John Mason, 2006). In the same paper the author advises "… there must be more to tasks than the overt or outer aspect: what learners are asked to do". In our main example of a less-same-more task we try to give some indication of how it can facilitate insights beyond merely the completion of the task. You will have this pleasure yourself if you work through any of the six further examples, including John Mason's original *More or Less Perimeter and Area* task which we have reproduced at the end of the article.

Task: Slam Dunk!

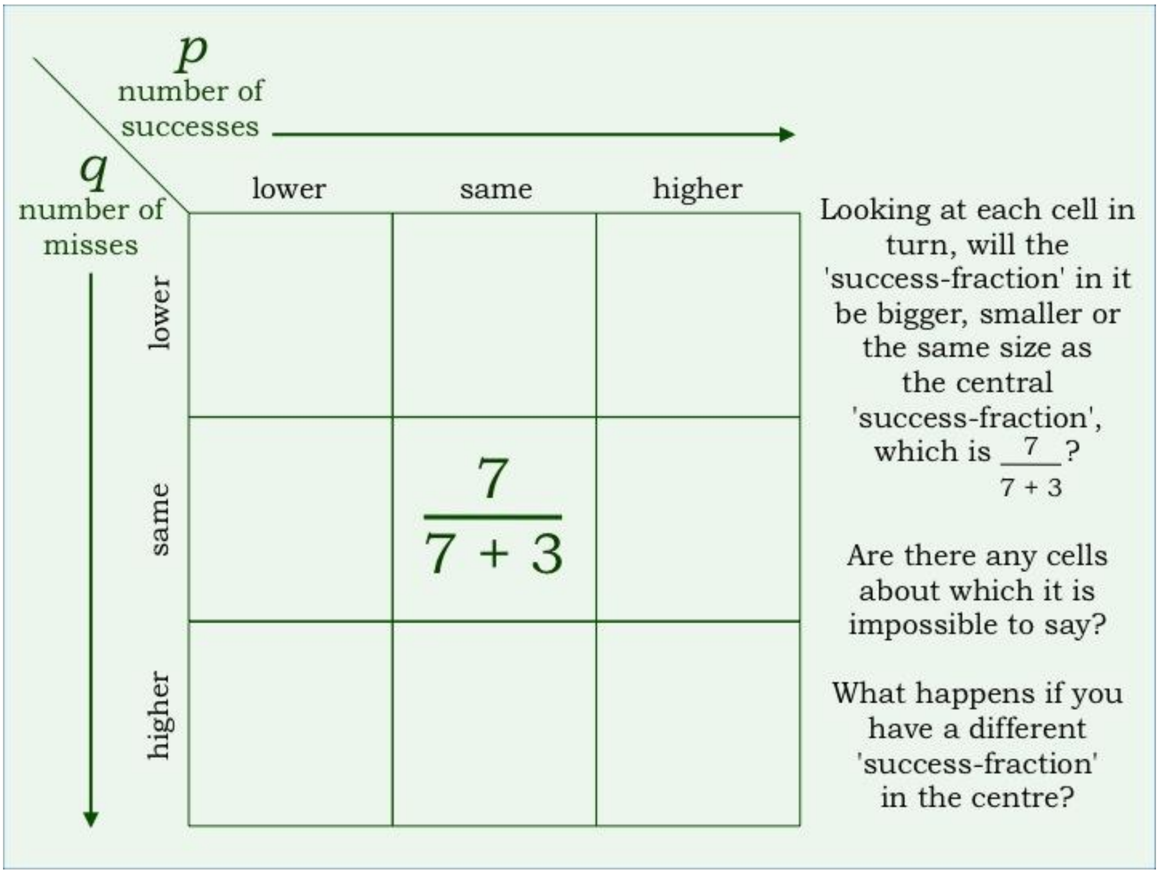
Pupils focus on the concepts of numerator, denominator and fraction-size. They first work with fractions of the form $p/(p + q)$, where p and q are whole numbers, before shifting attention to the simpler fraction form, p/q. The mathematical concept should be introduced as an exploration about a human situation, for example:

Basketball 'trials' are used to select players for a team.
In one of the trials players hoping to be in the team try to throw a ball through the hoop. They can have as many 'shots' as they like.

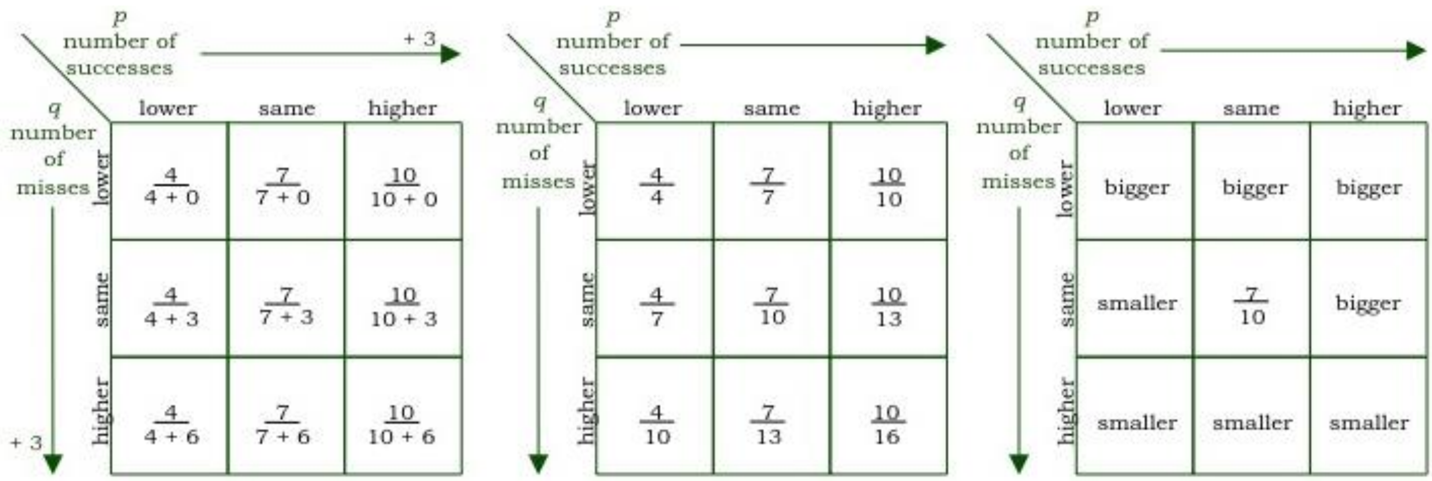
p is the number of times the player gets the ball through the hoop.
q is the number of times the player misses.

$\frac{p}{p+q}$ $\left(\frac{\text{number of successes}}{\text{number of 'goes'}}\right)$ is the player's hoop-scoring '***success-fraction***'.

A player's 'success-fraction' is one way of predicting how successful he or she will be as a team player. A success-fraction of $\frac{9}{10}$ is more likely to win the person a place in the team than a success-fraction of $\frac{4}{10}$.



Pupils are likely to decide that, in order to get a sense of what happens, they need to look at one example – at least! For example, they might increase/decrease the values of *p* and *q* given in the central fraction by 3 …



On the basis of this one example a pupil might make conjectures such as …

- if the number of successes is the same (as in the central fraction), the 'success-fraction' is bigger when the number of failures is less, but smaller when the number of failures is greater. (Is this 'common sense'?)
- if the number of failures is greater (than in the central fraction), the 'success-fraction' is smaller – whatever the number of successes is. (Is this 'common sense?)

- if both the number of successes and the number of failures are greater, the 'success-fraction' will be smaller. (Again, does this seem correct – is it best for a player to choose to have only a very few attempts?)

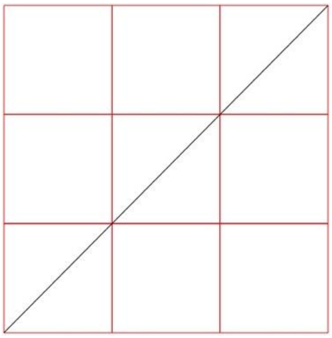
In order to get a better idea about the validity of what has been conjectured after looking at only one example pupils should feel the need to examine more possibilities with 7/(7 + 3) as the central fraction, and then to look at examples with other values of p and q in the central cell. For example, if they substitute $p = 100$, $q = 2$, and increase or decrease both p and q by 1 from cell to adjacent cell, they will see that the fraction in each cell, compared with the central fraction is …

bigger	bigger	bigger
smaller	$\frac{p}{p+q}$	bigger
smaller	smaller	smaller

whereas if they substitute $p = 2$, $q = 100$, and again increase or decrease both p and q by 1 from cell to adjacent cell, they will see

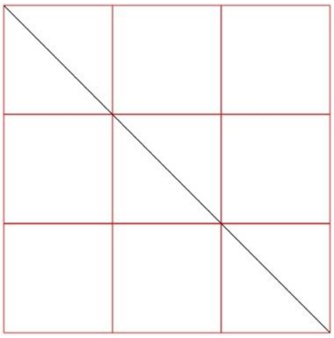
smaller	bigger	bigger
smaller	$\frac{p}{p+q}$	bigger
smaller	smaller	bigger

As pupils try more and more of their own examples, probably at first chosen without much reasoning behind the choices, there will be plenty for them to notice. For example, if they increase or decrease both p and q by the same amount from cell to adjacent cell, the denominators of the fractions on the SE-NW diagonal



will be equal because $(p - k) + (q + k) = (p + q) = (p + k) + (q - k)$. So, whatever the values of p and q are, these three fractions are always easy to compare: they should be able to reason that, since $p + k > p$, the fraction in the top-right cell ***must*** be bigger than the central fraction; and that, since $p - k < p$, the fraction in the bottom-left cell ***must*** be smaller. In the context, this means that if the number of times that a player gets the ball in the basket is greater and the number of failures is less, then the basketball 'success-fraction' is bigger – in line with common sense! But if the number of successes is less while the number of failures is greater, the 'success-fraction' is smaller – again common sense!]

When they look at the fractions on the NW-SE diagonal



pupils are likely to notice that both the top-left and bottom-right cells contain fractions that are bigger than the central fraction for some values of *p* and *q*, but that are smaller for other values. So the question that arises naturally from THEIR investigations is: “For what values (if any) of *p* and *q* will the fractions in the top-left and bottom-right cells both be equal to the central fraction?”. In the context of basketball ‘success-fractions’ this question could be posed as “Suppose that Steve gets k more ‘hoops’ and misses k more times than Robert does, and Robert gets k more ‘hoops’ and misses k more times than Charlie, but all three contestants have the same ‘success-fraction’, what can be deduced about Robert’s numbers of hoops and misses?”

Pupils need the practice of choosing examples intelligently: they need to learn how to pick-out ‘significant’ examples – a lesson learned, often, after experience of not doing so! In the paper cited earlier John Mason writes “Paulo Boero (2001) drew attention to the vital role of anticipation in mathematics … You do not embark on random calculations; rather you anticipate something and then check it out.” Since the question about equality of the fractions on the leading diagonal involves contemplating a symmetrical arrangement of ‘smaller, ‘bigger’ and ‘equal’

equal	bigger	bigger
smaller	equal	bigger
smaller	smaller	equal

and since the central fraction itself, $p/(p + q)$, is symmetrical when written in this form only if $p = q$, a pupil might anticipate that the fractions on the leading diagonal will be equal when $p = q$, and so choose to explore an example in which $p = q$ and both *p* and *q* are increased/decreased by the same constant number when moving from cell-to-adjacent-cell, such as

Value of q ↓ / Value of p →	lower	same	higher
lower	$\frac{1}{2}$ $\frac{9}{9+9}$ $\frac{9}{18}$	$\frac{10}{10+9}$ $\frac{10}{19}$	$\frac{11}{11+9}$ $\frac{11}{20}$
same	$\frac{9}{9+10}$ $\frac{9}{19}$	$\frac{1}{2}$ $\frac{10}{10+10}$ $\frac{10}{20}$	$\frac{11}{11+10}$ $\frac{11}{21}$
higher	$\frac{9}{9+11}$ $\frac{9}{20}$	$\frac{10}{10+11}$ $\frac{10}{21}$	$\frac{1}{2}$ $\frac{11}{11+11}$ $\frac{11}{22}$

The conjecture that the top-left and bottom-right fractions are equal when, and only when, $p = q$, is reached by looking at examples (such as the previous one). Some pupils will want to prove this by reasoning algebraically, and we have included both arguments ("if" and "only if") in the Appendix to this article. In the context of the basketball situation this is saying that "If Steve gets k more 'hoops' and misses k more times than Robert, and Robert gets k more 'hoops' and misses k more times than Charlie, and all three contestants have the same 'success-fraction', then Robert and Steve and Charlie all get the ball into the basket on exactly half of whatever is their total number of goes!"

This might prompt pupils to wonder whether the smaller/bigger alternatives in the leading diagonal corners depend (crucially) on whether $p < q$ or $p > q$. If they choose examples to check this out pupils will obtain support for the conjecture that:

If $\frac{p}{q} < 1$ then

smaller		
		bigger

If $\frac{p}{q} > 1$ then

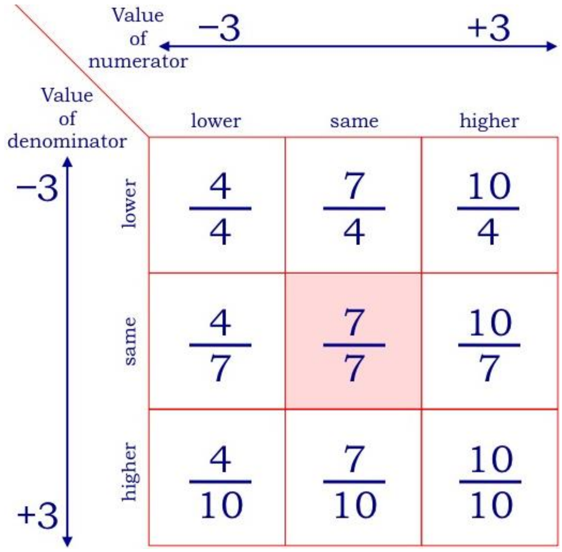
bigger		
		smaller

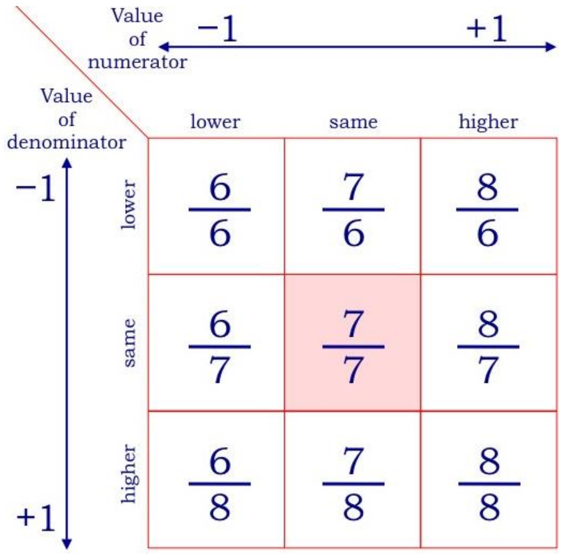
If $\frac{p}{q} = 1$ then

equal		
		equal

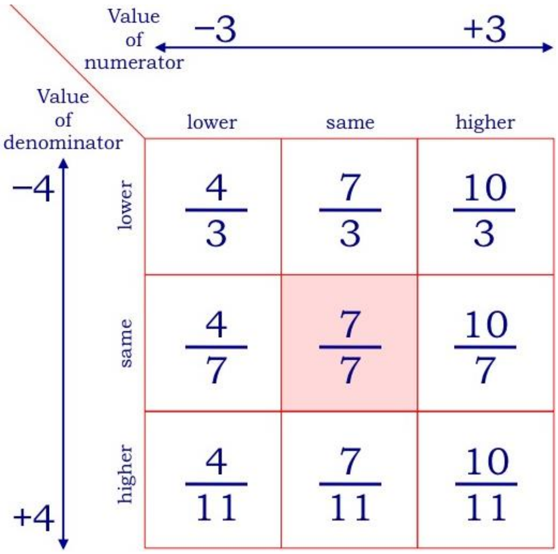
Having begun to get some idea of what happens to the size of fractions of the form $p/(p + q)$ as p and q are increased or decreased in systematic ways (such as interpreting 'increasing or decreasing' strictly as 'adding or subtracting a constant number'), pupils might feel that they would gain a better idea of how to proceed to a more general understanding if they first looked at what happens with the simplest fraction-form, p/q.

When pupils look at a few examples, they will find, as before, that if p and q are increased or decreased by the same amount, the fractions on the leading diagonal are equal when, and only when, $p = q$. The algebraic argument is in the Appendix.





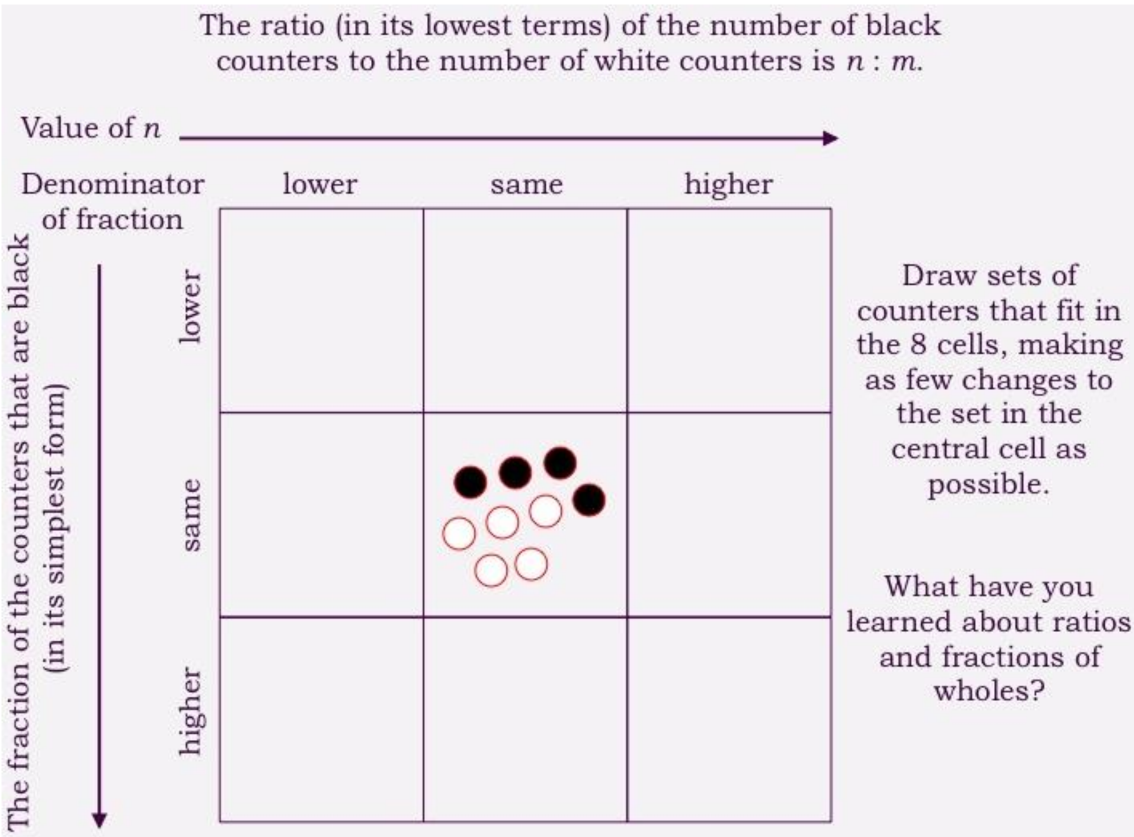
so they may wonder what happens if the values of p and q increase/decrease by different amounts: will it still be true that the fractions on the leading diagonal are equal only when $p = q$? With just a little exploration of any numerical example

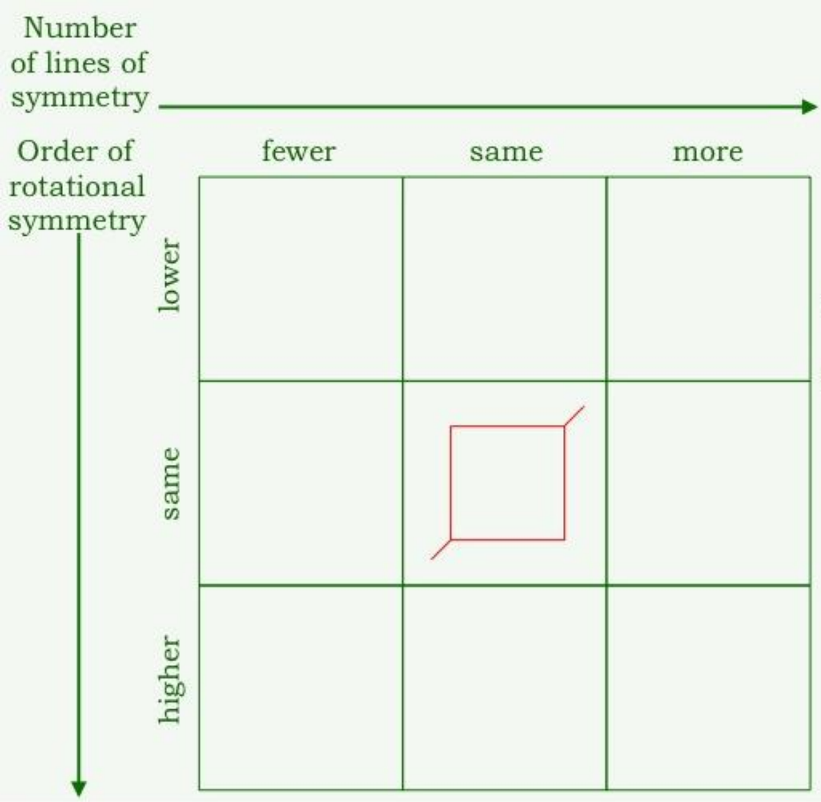


pupils should be able to explain why this is not the case.

Further examples

Same-less-more tasks exemplify the principles quoted at the start of this article, and they can be constructed in many other mathematical contexts. Here are five examples that provide opportunities for pupils to develop their reasoning skills and deepen their understanding of various key mathematical concepts – without any need whatsoever to rush ahead in the textbook, scheme of work or key stage planner!

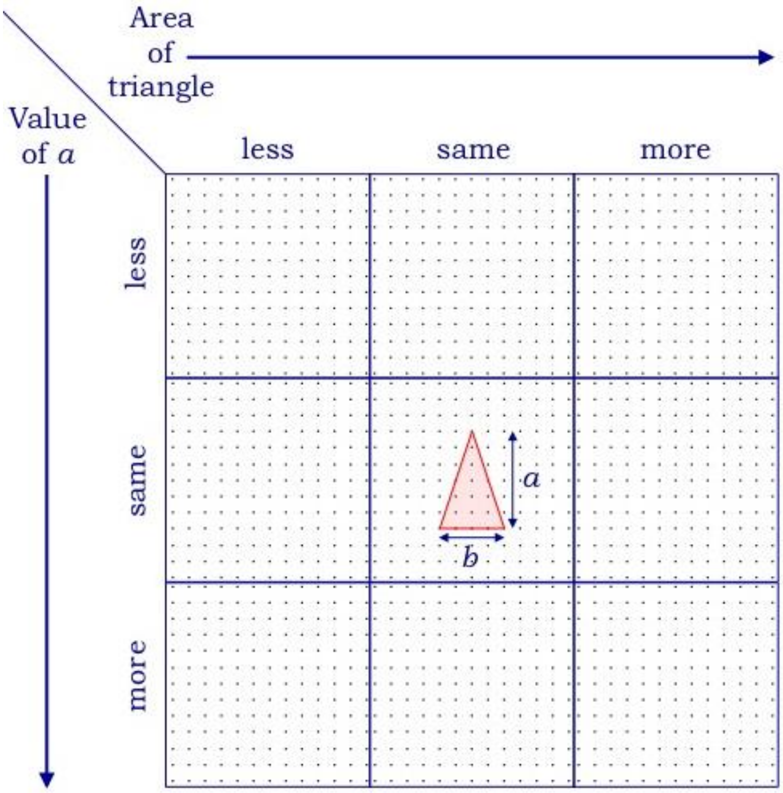




By copying the central image and adding line segments to it, try to construct images that fit in the 8 cells.

Are any cells impossible to fill in this way?

What have you learned about lines of symmetry and rotational symmetry?



The perpendicular distance from the side of length b to the opposite corner of the triangle is a.

In each of the 8 cells draw a triangle (with ALL its corners on dots) that fits the cell, making as little change to the central triangle as possible.

In the different cells, what is BOUND to happen to the value of b? Why? Is it possible to be certain for every cell? Why or why not?

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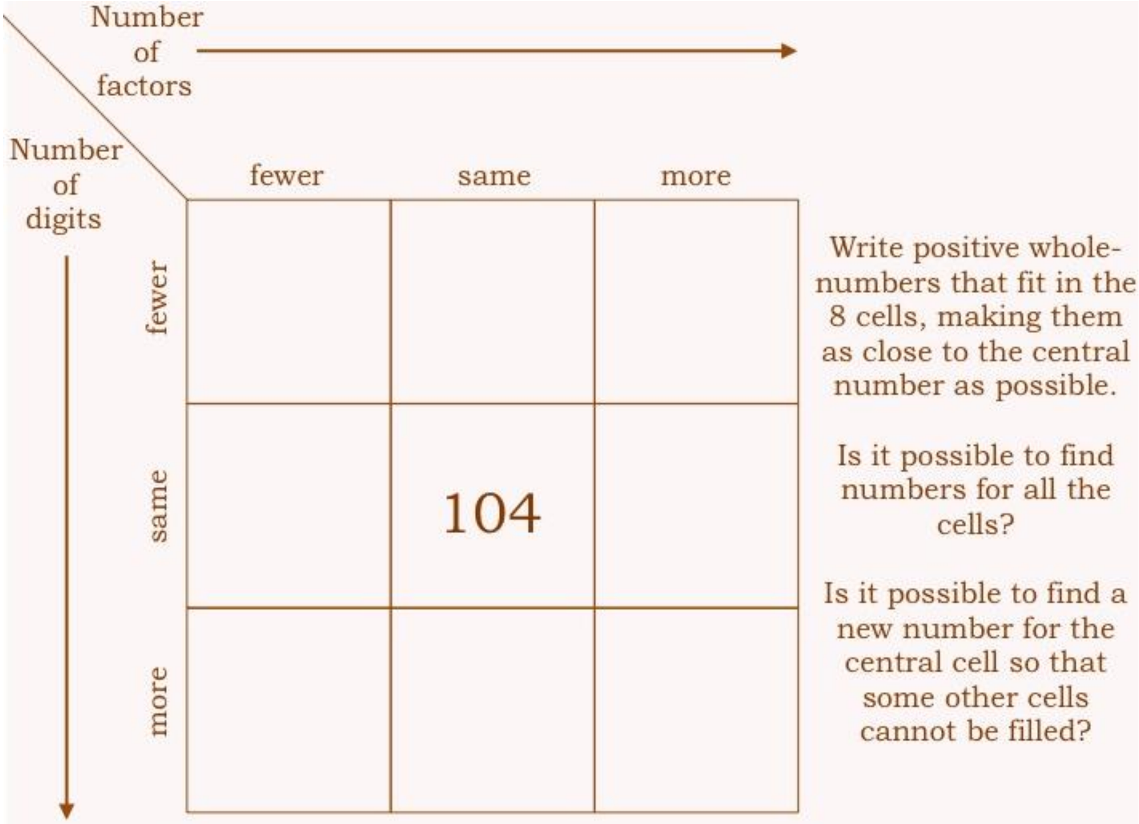

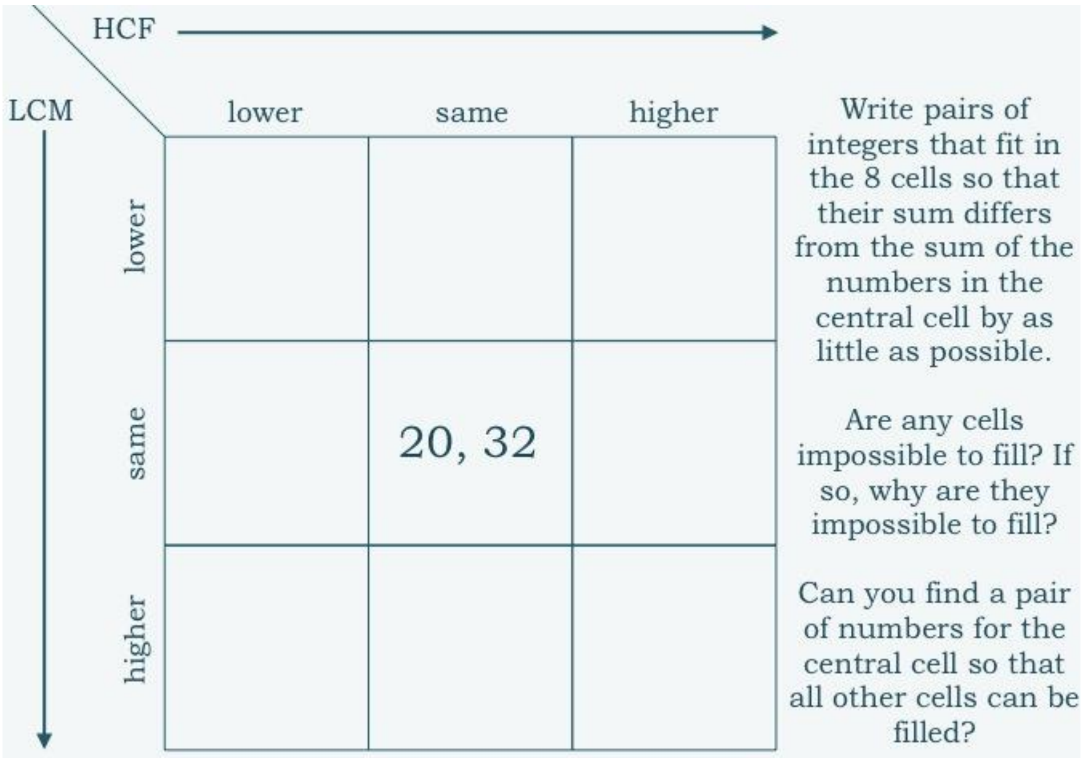

Our last example is John Mason's *More or Less Perimeter and Area* task, our inspiration for this article.

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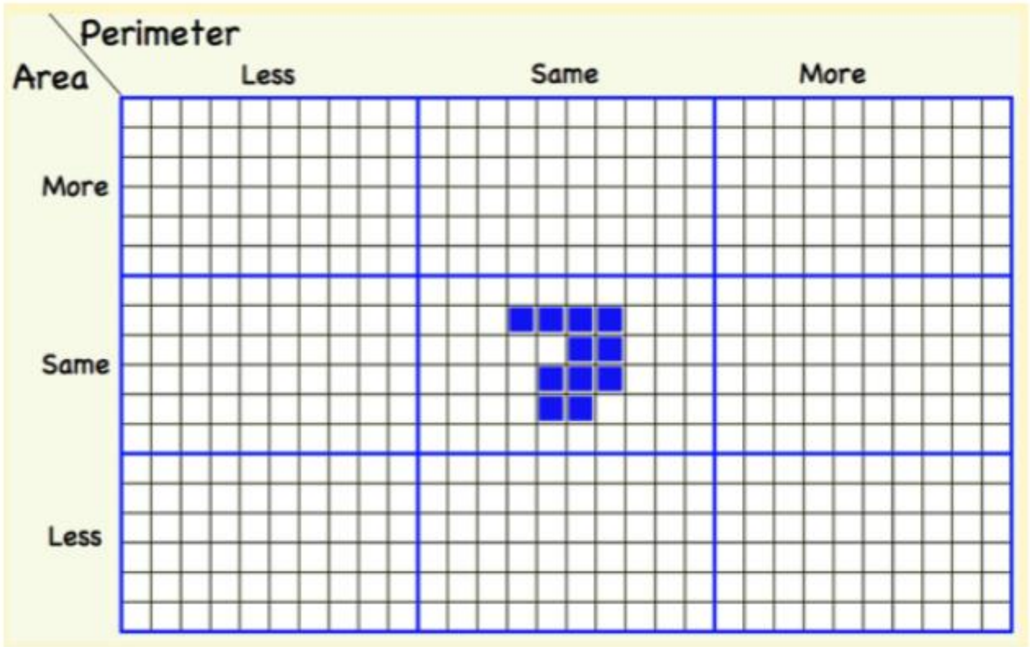
Perimeters and Areas

John Mason
ATM London 2015

Coordinated Perimeters

See ppt for animation, or applet

More or Less Perimeter and Area



Construct shapes that fit in the 8 cells, making as few changes to the central shape as possible.

Which cell was hardest to fill?

What have you learned about modifying shapes so as to change either the perimeter or the area but not both?

What other examples of less-same-more tasks occur to you? Have you tried something like this with your pupils? Let us know: email info@ncetm.org.uk, or tweet us @NCETM.

Appendix: algebraic arguments

Conjecture 1: the top-left and bottom-right fractions are equal if, and only if, $p = q$

Value of q ↓ \ Value of p →	lower	same	higher
lower	$\frac{(p-k)}{(p-k)+(q-k)}$	$\frac{p}{p+(q-k)}$	$\frac{(p+k)}{(p+k)+(q-k)}$
same	$\frac{(p-k)}{(p-k)+q}$	$\frac{p}{p+q}$	$\frac{(p+k)}{(p+k)+q}$
higher	$\frac{(p-k)}{(p-k)+(q+k)}$	$\frac{p}{p+(q+k)}$	$\frac{(p+k)}{(p+k)+(q+k)}$

A Department for Education initiative to enhance professional development across mathematics teaching

If $p = q$, then

$$\frac{(p-\text{k})}{(p-\text{k})+(q-\text{k})} = \frac{(p-\text{k})}{(p-\text{k})+(p-\text{k})} = \frac{(p-\text{k})}{2(p-\text{k})} = \frac{1}{2}$$

$$\frac{(p+\text{k})}{(p+\text{k})+(q+\text{k})} = \frac{(p+\text{k})}{(p+\text{k})+(p+\text{k})} = \frac{(p+\text{k})}{2(p+\text{k})} = \frac{1}{2}$$

$$\frac{p}{p+q} = \frac{p}{p+p} = \frac{p}{2p} = \frac{1}{2}$$

So $\dfrac{(p-\text{k})}{(p-\text{k})+(q-\text{k})} = \dfrac{(p+\text{k})}{(p+\text{k})+(q+\text{k})} = \dfrac{p}{p+q}$

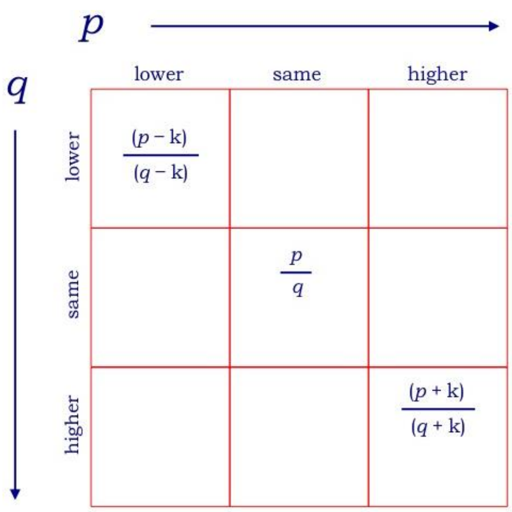
If

$$\frac{(p-\text{k})}{(p-\text{k})+(q-\text{k})} = \frac{p}{p+q} = \frac{(p+\text{k})}{(p+\text{k})+(q+\text{k})}$$

then

$(p-\text{k})(p+q)$	$=$	$p(p+q-2\text{k})$	$p(p+q+2\text{k})$	$=$	$(p+\text{k})(p+q)$
$p^2+pq-\text{k}p-\text{k}q$	$=$	$p^2+pq-2\text{k}p$	$p^2+pq+2\text{k}p$	$=$	$p^2+pq+\text{k}p+\text{k}q$
$-\text{k}p-\text{k}q$	$=$	$-2\text{k}p$	$2\text{k}p$	$=$	$\text{k}p+\text{k}q$
$-p-q$	$=$	$-2p$	$2p$	$=$	$p+q$
p	$=$	q	p	$=$	q

Conjecture 2: if p and q are increased or decreased by the same amount, the fractions on the leading diagonal are equal if, and only if, $p = q$



If

$$\frac{(p-k)}{(q-k)} = \frac{p}{q} = \frac{(p+k)}{(q+k)}$$

then

$$\begin{aligned} q(p-k) &= p(q-k) \\ pq - kq &= pq - kp \\ -kq &= -kp \\ q &= p \end{aligned} \qquad\qquad \begin{aligned} p(q+k) &= q(p+k) \\ pq + kp &= pq + kq \\ kp &= kq \\ p &= q \end{aligned}$$

If

$$p = q$$

then

$$\frac{(p-k)}{(q-k)} = \frac{(p-k)}{(p-k)} = 1 \qquad \frac{(p+k)}{(q+k)} = \frac{(p+k)}{(p+k)} = 1 \qquad \frac{p}{q} = \frac{p}{p} = 1$$

So

$$\frac{(p-k)}{(q-k)} = \frac{p}{q} = \frac{(p+k)}{(q+k)}$$

You can find previous *It Stands to Reason* features here

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Secondary Magazine 131

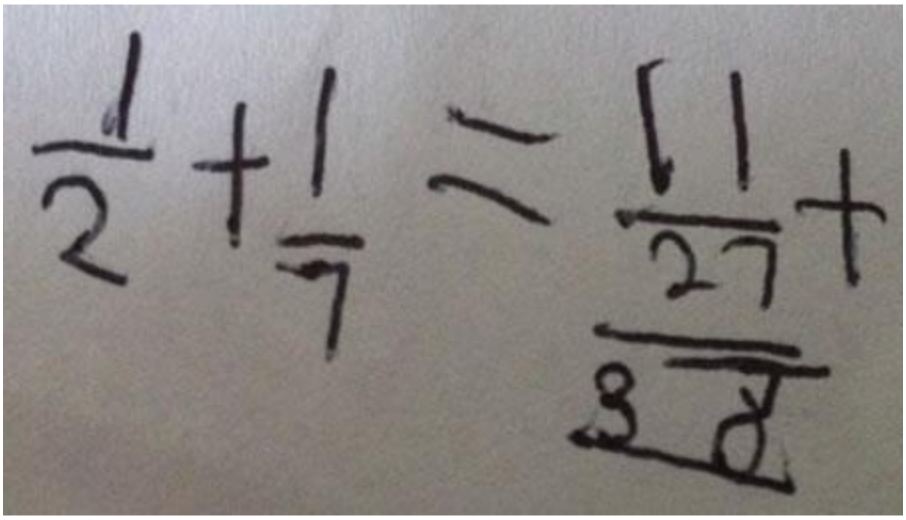
National Centre for Excellence in the Teaching of Mathematics

Eyes Down

by Lisa Woods, a secondary school maths teacher working in Yorkshire

This isn't from one of my students, but from my daughter's homework. She is in Year 7 (at a different mainstream school from the one where I teach), has autism and finds maths particularly challenging.

The task she'd been given was to work out $^1/_2 + {}^1/_7$, and she proudly came and told me the answer was $^3/_8$ (not quite the $^2/_9$ I was expecting), so I asked her how she worked it out and this was what she showed me:



This made it clear to me that my daughter has been shown a number of methods for responding to different maths problems but she is not sure which to apply in the different scenarios. Of course, we see this right across the secondary age and attainment spectrum – think of the A/A* candidates who simplify correctly $(x^2 – 5x + 6) ÷ (x^2 + x – 12)$ to get $(x – 2) ÷ (x + 4)$ … and then cross out the x's and declare that the answer is -0.5!

My daughter struggles with the concepts of money and the 24 hour clock (time generally). I am pleased with the work both her primary did and secondary school are doing to support her, but she does not seem to have any innate grasp of numbers. When she was at primary school I printed off the numbers 1 to 10 on little cards for her to practise ordering them. She managed it after about 6 months of practice, and then asked me "I wonder how many cards there are?" ... so we counted them ... got to 10 … and I waited for the lightbulb moment that never happened: it seems that, to her, asking "what is purple + red?" makes as much sense as "what is 5 + 3?"

There are no easy answers to this I guess, but we sometimes need to remind ourselves that not all students are created equal – either in the precise sense of their cognitive architecture, or more broadly in the sense of their mathematical experiences and learning prior to joining our classes.

If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb) to us at info@ncetm.org.uk with 'Secondary Magazine Eyes Down' in the email subject line. Include a note of where and when it was taken, and any comments on it you may have. If your picture is published, we'll send you a £20 voucher.

You can find previous *Eyes Down* features here

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